Electrodynamics in one dimension: radiation and reflection

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Abstract
Problems involving polarized plane waves and currents on sheets perpendicular to the wavevector involve only one component of the fields, so it is possible to discuss electrodynamics in one dimension. Taking for simplicity linearly polarized sinusoidal waves, we can derive the field emitted by currents (analogous to dipole radiation in three dimensions) and reflection and transmission (analogous to Thomson scattering). Some aspects of the results are not intuitive, and for a physical understanding we need to see the problem in three dimensions. Eventually we apply these results to a linear model of the sheet, and we discuss the limit from a thick sheet.

1. Introduction

All textbooks on electromagnetism include the description of a linearly polarized plane wave, where only one spatial dimension is involved (see Griffiths (1999), p 364). We want to complete this description by including also the charges, discussing the case of a delta-function charge distribution, that is, a uniform sheet of an infinitely thin medium with charge carriers (in other words, the Green’s function of one-dimensional (1D) electrodynamics). This is an elementary case, suitable also for the undergraduate level, from which any 1D problem can be constructed, but in itself interesting in some practical cases.

The elementary case we want to discuss is the reflection of a linearly polarized sinusoidal wave on an infinitely thin charge distribution, with positive and negative charges with the same density, but one of them fixed (so there is no longitudinal field).

The results have some surprising, non-intuitive aspects, and for this reason it is necessary to analyse them in detail, and we find that these phenomena are understood if the problem is considered in its three dimensions, taking into account the effects of retardation due to the light velocity.

In particular the electric field $E$ produced near the sheet turns out to be in phase with the velocity of the charges and not with its acceleration as one would expect (and happens
for a point charge). But actually one has to take into account the effect of the other charges contained within a circle expanding at the speed of light.

In the second part of the paper (section 3) as an application we consider a phenomenological model including resistive and dielectric properties of the sheet, finding the reflection and transmission coefficients and showing that the results are very easily obtained and agree with the expressions reported in the literature for thick sheets going to the limit of zero thickness. We also find that for a sheet with zero resistance we have total reflection independently of the surface charge density. This part is more specialized and is suitable for graduate level.

2. 1D Maxwell equations

We take a uniform sheet of charge in the plane $x = 0$ (compensated by a static charge of opposite sign). Its motion produces electromagnetic fields which are only a function of $x$ (and $t$). For simplicity we consider the motion of charges only along $z$ and a surface density $N^*$ of carriers of charge $e$, moving with velocity $v$ (see figure 1). They give rise to a uniform surface current $\tau = N^*ev$. $E$ is directed only along $z$ and $B$ along $y$, so we can drop indices $y$ and $z$.

Then $E$ and $B$ do not depend on $y$ and $z$ and the Maxwell equations reduce to

$$\begin{align*}
\frac{\partial E}{\partial z} &= 0, \quad \frac{\partial B}{\partial y} = 0, \\
\frac{\partial E}{\partial x} &= \frac{\partial B}{\partial t}, \quad \frac{\partial B}{\partial x} = \mu_0 N^* ev \delta(x) + \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}
\end{align*}$$

(1)

where $\delta(x)$ is a delta function.

Outside the charges, the solution is the well-known one for a plane linearly polarized wave propagating along $\pm x$.

2.1. Fields produced by a given current

Let us first find the fields for an assigned current $\tau = N^*ev(t)$. 

\[\text{Figure 1. Graphical representation of a current sheet of density } \tau, \text{ the field vectors } E \text{ and } B, \text{ and the Poynting vector } S.\]
At \( x = \pm 0 \) the fourth equation gives (see Jackson (1975), equation (1.20))

\[
B = \frac{1}{2} \mu_0 \tau(t) \frac{x}{|x|}.
\]

From the third Maxwell equation in vacuum, we obtain

\[
E = -\frac{1}{\omega \epsilon_0 \mu_0} k \times B = -\frac{c}{k} k \times B
\]

where \( k \) is the wave vector. In the case when \( k / k \) coincides with the unit vector of the \( x \)-axis, we have

\[
E = -c \frac{x}{|x|} \hat{x} \times \frac{x}{|x|} B = -c \hat{x} \times \hat{y} \mu_0 \tau(t) - \frac{c \hat{z} \mu_0 \tau(t)}{2}.
\]

This is surprising, as we know that an accelerated charge produces an electric field proportional to acceleration, not to velocity. But because of retardation, the vector potential \( A \) is parallel and proportional to the integral of current inside a circle of radius \( ct \) on the plane \( x = 0 \). This circle is expanding at speed \( c \) from the origin starting at \( t = 0 \). The points outside this circle do not contribute to \( A \) because they ought to be considered at a time before \( t = 0 \). At each instant, during the interval \( dt \), an increment \( dA \) comes from the area of a circular crown of radius \( ct \) and thickness \( c dt \). But this contribution is also inversely proportional to the distance \( ct \), and therefore \( A \) turns out to be proportional to \( t \); as a consequence, the resultant electric field \( E = -\partial A/\partial t \) is merely proportional to the current and opposite to it.

Now we expect that the field generated at \( x = \pm 0 \) merely propagates along \( +x \) and \( -x \) with velocity \( c \). Let us see it in detail with a direct calculation for a current sinusoidal in time:

\[
\tau(t) = \tau_0 \sin \omega t.
\]

Neutrality of the sheet allows us to take the scalar potential \( V = 0 \), and the vector potential, which is in the direction \( z \), is

\[
A(x,t) = \int_0^\infty \frac{\mu_0 \tau(t_r)}{4\pi} \frac{d\tau}{r} dz dy
\]

where \( r \) is the distance from the source point to the field point and \( t_r = t - r/c \) is the retarded time.

Then we obtain

\[
A(x,t) = \frac{\mu_0}{4\pi} T \int_0^\infty 2\pi R dR \frac{\sin[\omega(t - \frac{r}{c})]}{r} = \frac{\mu_0}{2} T \int_0^\infty d\tau \sin[\omega \left( t - \frac{r}{c} \right)]
\]

with \( R = \sqrt{r^2 - x^2} \). We must admit that before a certain time in the past the current was zero; otherwise the integral would be undetermined. Under these conditions the integral can be calculated giving

\[
A(x,t) = \frac{\mu_0 c}{2\omega} T \cos\left( \frac{\omega |x|}{c} - \omega t \right).
\]

Hence the electric field turns out to be

\[
E(x,t) = -\frac{\mu_0 c}{2} T \sin\left( \omega \left( t - \frac{|x|}{c} \right) \right).
\]

Note that on the \( yz \) plane, \( E(x,t) \) has opposite phase to \( \tau(t) \). These remarks show that, although everything can be calculated by considering the 1D problem and using equations (1), for a physical understanding of the interactions one has to consider the three-dimensional picture. A similar problem is raised by the textbook of Feynman et al (1964) and discussed by several other authors (Abbott and Griffiths 1985, Peters 1986, Holstein 1995, Chung 2008): a current is suddenly switched on; the emitted field propagates and contains
more and more energy: how can this be provided by a sheet in uniform motion? The answer, also in this case, lies in the retardation.

2.2. Reflection and transmission. Energy balance

For a better understanding of the behaviour of our current sheet with regard to important phenomena such as reflection and transmission, we have to calculate the relevant parameters including the impedance. Before that, however, concerning the energy balance, we can verify it without making hypotheses about the phase relationship between the incoming field $E_I$ and the emitted one $E$. Let the incoming electromagnetic plane wave propagate along the $x$-axis, polarized parallel to the $z$-axis:

$$E_I = \varphi_I \cos(kx - \omega t + \varphi_I) = \text{Re}(\tilde{E}_I)$$

(6)

with

$$\tilde{E}_I = \varphi_I \exp(ikx - i\omega t)$$

and

$$\varphi_I = \varphi_I \exp(i\varphi_I).$$

Its electric field at the location of our sheet is obtained by taking $x = 0$ in equation (6).

We can now examine the energy balance on the basis of these essential hypotheses in our system. The incoming intensity is

$$I = \langle S_I \rangle = \frac{E_I^2}{2Z_0}$$

(7)

corresponding to an instantaneous surface power density

$$S_I = E_I^2 / Z_0 = \frac{1}{Z_0} \varphi_I^2 \cos^2(-\omega t + \varphi_I)$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the vacuum impedance.

The outgoing instantaneous power density is given by the sum of the following terms:

- $(E_I + E)^2 / Z_0 = (E_I + E)^2 / Z_0$, the power transmitted beyond the sheet (the field beyond the sheet is the sum of incoming and radiated fields);
- $E_I^2 / Z_0$, the reflected power and
- $(E_I + E)\tau$ the power yield to the charges on the sheet.

Summing over these three terms and taking into account that

$$\tau = -2E / Z_0$$

(8)

we obtain

$$\frac{1}{Z_0}[(E_I + E)^2 + E^2 - 2(E_I + E)E] = E_I^2 / Z_0$$

(9)

thus verifying the energy conservation. It is worth noting that the energy balance is guaranteed whatever is the phase relationship between the total field acting on the electric charges, $E_I + E$, and current $\tau$.

Considering average values we have constant reflection and transmission coefficients which can be expressed in terms of the characteristic impedances.

We define the following:

the amplitude of the reflection coefficient $R = \varphi / \varphi_I$,

the amplitude of the transmission coefficient $T = (\varphi_I + \varphi) / \varphi_I$ and
the surface impedance of the sheet \( Z = (\tilde{\phi}_1 + \tilde{\phi})/\tau_0 \), where the denominator indicates the complex amplitude of the surface current density. Note that the transmitted amplitude is the sum of the field produced by the sheet and the incoming one.

From equation (8) and the above definitions, we obtain

\[
R = \frac{-1}{1 + 2Z/Z_0} , \quad T = \frac{2Z/Z_0}{1 + 2Z/Z_0} .
\]

Note that the same result can be obtained by the usual expressions

\[
R = \frac{Z_L - Z_0}{Z_L + Z_0} , \quad T = \frac{2Z_L}{Z_L + Z_0} ,
\]

for the reflection and transmission coefficients of a wave encountering a load impedance \( Z_L \) which is given by the parallel of the sheet impedance \( Z \) and the vacuum impedance \( Z_0 \) beyond it.

Note that for a ‘perfect’ conductor (\( Z = 0 \)), we have total reflection independently of the surface charge density.

The energy balance can be written in terms of the fractional powers

\[
W_R/W_I = |R|^2 \quad \text{and} \quad W_T/W_I = 1 - |R|^2
\]

3. A phenomenological model for the sheet

3.1. General considerations

We can now introduce the relation between the effective electric field and the current density. It represents, in a phenomenological fashion, the physical characteristics of the medium with regard to the mobility of the electric charge carriers, their effective mass and/or their bonds with the lattice. So let us assume that the charge carriers obey a general equation of motion including their mass, a damping term describing the energy dissipation and a harmonic oscillator term that accounts for the dielectric properties in the case of bonded charges (with only one resonance, for simplicity, i.e. one anomalous absorption band):

\[
e(E\dot{I} + E) = m\ddot{z} + m\gamma \dot{z} + m\omega_0^2 z
\]

where \( e \) is the charge of the charge carriers, \( m \) is their effective mass and \( \gamma \) is the damping coefficient.

Using the direct relation we have between the excited current and the emitted radiation,

\[
\tau = -2E/Z_0 \quad \text{i.e.} \quad N^*e\dot{z} = -2E/Z_0 ,
\]

we can directly calculate the ratio \( E/E_I \) from which we obtain immediately the reflectance of our sheet, or we can express our model in terms of the sheet impedance:

\[
Z = (\tilde{\phi}_1 + \tilde{\phi})/\tau = \frac{m}{N^*e^2} \left[ \gamma + i \left( \omega - \omega_0^2 \right) \right] .
\]

Let us consider each term separately.

First, free charges with mass \( m \) and surface density \( N^* \):

\[
Z = im\omega/(N^*e^2);
\]

then we have

\[
Z/Z_0 = im\omega/(N^*e^2Z_0)
\]
and for
\[ \omega \ll N^*e^2Z_0/m, \]
we have total reflection \((R = -1, i.e. \text{opposite phase to the incoming field; velocity proportional to } E, \text{ and emitted field proportional to the opposite of the velocity}), \text{ independently of the charge density. In contrast for} \]
\[ \omega \gg N^*e^2Z_0/m, \]
we obtain
\[ R = iZ_0N^*e^2/(2m\omega) \]
which means that the reflection amplitude tends to zero and is in quadrature with the incoming field; while \(T\) tends to 1.

Then, let us consider the resistive and the dielectric terms, and the limit from the case of a thick sheet.

3.2. Conducting sheet and the limit from a thick sheet

Let us consider first the case of a metal conductor such that we have only the central term on the right-hand side of equation (10). Taking account of equation (11), we obtain
\[ (\tilde{\chi}_I + \tilde{\chi}) = -\frac{2\tilde{\chi}}{Z_0\sigma a} \]
having expressed the damping coefficient in terms of conductivity \(\sigma\) and the sheet thickness \(a\), i.e.
\[ \gamma = N^*e^2/(a\sigma m). \]
Hence we obtain
\[ R = -\frac{\tilde{\chi}}{\tilde{\chi}_I} = -\left(1 + \frac{2}{Z_0\sigma a}\right)^{-1}. \]

The same result could be obtained by the standard procedure, i.e. utilizing the expression of the reflection coefficient in terms of the characteristic impedance of the two media \(Z_0\) and \(Z\) and combining the effects of the reflections on the two surfaces of the sheet. In this way the obtained reflection coefficient of a metal sheet of thickness \(a\) turns out to be (Hallen 1962, p 420)
\[ R = \frac{\tilde{\chi}_I}{\tilde{\chi}_I} = \frac{-(Z_0 + Z)(Z_0 - Z)e^{\gamma a} + (Z_0 + Z)(Z_0 - Z)e^{-\gamma a}}{(Z_0 + Z)(Z_0 + Z)e^{\gamma a} + (Z_0 - Z)(Z_0 - Z)e^{-\gamma a}} \]
where \(Z = \sqrt{i\omega\mu\varepsilon_0}/\sigma\) and \(\gamma = \sqrt{i\omega\mu\varepsilon_0}\) is the wave number in the 'low-frequency' limit, that is, for \(\omega\varepsilon\varepsilon_0 \ll \sigma\). For small thickness, i.e. for \(a \ll 1/\gamma\), (15) becomes
\[ R = -\frac{\tilde{\chi}_I}{\tilde{\chi}_I} = \frac{(Z_0^2 - Z^2)\gamma a}{2Z_0Z + (Z_0^2 + Z^2)\gamma a}. \]
In these conditions, and supposing \(\mu\) and \(\varepsilon\) of the order of unity, we can neglect \(Z\) with respect to \(Z_0\) so that
\[ R = -\frac{1}{1 + 2Z/(Z_0\gamma a)} = -\frac{1}{1 + 2/(Z_0\sigma a)} \]
3.3. Dielectric sheet and the limit from a thick sheet

In the case of a dielectric sheet at normal optical frequencies (far from absorption bands), we can retain only the last term on the right-hand side of equation (10) obtaining

\[ R = -\frac{\tilde{X}}{\tilde{X}_I} = -\frac{1}{1 - \frac{i 2}{a(\varepsilon_r - 1)} k_0} \]  

(18)

where we have expressed the coefficient of \( z \) in terms of the dielectric constant, i.e.

\[ m_0^2 = \frac{N^* e^2}{a \varepsilon_0 (\varepsilon_r - 1)}. \]

Because of the fact that \( ak \ll 1 \), we can neglect the unity in the denominator obtaining

\[ R = -\frac{\tilde{X}}{\tilde{X}_I} = -\frac{i 1}{2} a(\varepsilon_r - 1) k_0. \]

This result is in agreement with the expression of the reflection coefficient for a thin dielectric layer reported by Ramo et al (1994, p 293) assuming, as usual, \( \mu_r = 1 \)

\[ R = \frac{1}{2} k_0 \left( \frac{Z}{Z_0} - \frac{Z_0}{Z} \right) = \frac{1}{2} a(\mu_r - \varepsilon_r) k_0 = \frac{1}{2} a(1 - \varepsilon_r) k_0. \]

We remark that the formalism of impedances allows us not to take explicitly into account multiple reflections (Schelkunoff 1938, Pereyra and Robledo-Martinez 2009).

4. Conclusions

The main purpose of this paper is to analyse the one-dimensional problem of the interaction of a plane sinusoidal electromagnetic wave with a thin sheet of charges and to discuss the relationship between the incoming field, the field acting on the sheet charges and the reflected and transmitted amplitudes and powers, and to show that although the problem is one dimensional (and can be solved in one dimension), in order to understand the physical phenomena involved, it is necessary to see it in three dimensions and take into account the retardation due to light velocity. We can say that the field emitted by an assigned current is the 1D equivalent of the emission from a point dipole, and the reflection coefficient corresponds, for free charges, to Thomson scattering. In 1D it is all more simple.

The expressions found may also be useful for obtaining in a straightforward way useful parameters such as impedance and reflection coefficients for cases of practical importance, when the layer thickness can be considered negligible with respect to the wavelength of the incident radiation.

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