The Ward-Takahashi Identity in Supersymmetric lattice models

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International Workshop on "Actions and symmetries in lattice gauge theory"

Yukawa Hall, Yukawa Institute for Theoretical Physics (YITP), Kyoto University

February 13-26, 2006

Outline

1. $N = 1$ Super Yang-Mills Theory with Wilson Fermions: study of the Ward-Takahashi identity

2. Exact formulation of lattice supersymmetry for the $N = 1$ Wess-Zumino model in 4 dimensions using the Ginsparg-Wilson operator

The Model

The continuum action of $N = 1$ SYM and gauge group $SU(N_c)$ reads

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} \bar{\lambda}^a \gamma_\mu (\mathcal{D}_\mu \lambda)^a + m_{\tilde{g}} \bar{\lambda}^a \lambda^a \,,$$

$\lambda = \lambda^a T^a$ is a Majorana spinor in the adjoint representation of the gauge group that satisfies the Majorana condition

$$\bar{\lambda} = \lambda^T C \,, \qquad \lambda = C \bar{\lambda}^T$$

The gluon fields are represented by

$$\begin{aligned} A_\mu &= -ig A^a_\mu T^a \\ F_{\mu\nu} &= -ig F^a_{\mu\nu} T^a \\ \mathcal{D}_\mu \lambda^a &= \partial_\mu \lambda^a + g f_{abc} A^b_\mu \lambda^c \,. \end{aligned}$$

For $m_{\tilde{g}} = 0$ has a supersymmetry respect to the SUSY transformations.

The continuum SUSY transformations read

$$\begin{aligned} \delta A_\mu(x) &= -2g\bar{\lambda}(x)\gamma_\mu\varepsilon \\ \delta\lambda(x) &= -\frac{i}{g}\sigma_{\rho\tau}F_{\rho\tau}(x)\varepsilon \\ \delta\bar{\lambda}(x) &= \frac{i}{g}\bar{\varepsilon}\sigma_{\rho\tau}F_{\rho\tau}(x) \end{aligned}$$

where $\sigma_{\rho\tau} = \frac{i}{2}[\gamma_\rho, \gamma_\tau]$ and ε is a global Grassmann parameter with Majorana properties.

These transformations relate fermions and bosons.

They leave the action invariant and commute with the gauge transformations so that the resulting Noether current $S_\mu(x)$ is gauge invariant.

For $N = 1$ SYM theory the supercurrent is

$$S_\mu = -F^a_{\rho\tau}\sigma_{\rho\tau}\gamma_\mu\lambda^a\,.$$

Classically the Noether theorem is conserved

$$\partial_\mu S_\mu = 0\,,$$

(if the fields satisfy the eq. of motion). Furthermore, it fulfills a spin 3/2 constraint

$$\gamma_\mu S_\mu = 0\,.$$

SUSY WIs

The existence of the renormalized supercurrent S_μ^R is assumed (where $S_\mu^R = Z_S S_\mu + Z_T T_\mu$)

$$\partial_\mu S_\mu^R = 2 m_R \chi_R$$

where

$$\chi_R = Z_\chi \chi \,, \qquad \chi \equiv \frac{1}{2} F_{\mu\nu}^a \sigma_{\mu\nu} \lambda^a \,.$$

m_R is the renormalized gluino mass.

- SUSY occurs for $m_R = 0$.
- The non-vanishing of m_R describes a **soft breaking** of SUSY.

Lattice formulation of SYM theory

- The lattice regularized theory is not SUSY as the Poincaré invariance (a sector of the superalgebra) is lost.

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu$$

  Not a severe problem. Calculating at several lattice spacing a and then take the limit $a \to 0$. No fine tuning is needed.

- If there are scalar mass terms in the SUSY theory that break SUSY. Since this operators are relevant fine tuning is necessary to cancel their contributions.

- A naive regularization of fermions results in the doubling problem

  Nielsen & Ninomiya 1981

  $\rightarrow$ wrong number of fermions and violation of the balance between bosons and fermions

  – The problem can be treated as in QCD using "traditional" fermions

Fermions on the Lattice

- Recently, following the rediscovery of the Ginsparg-Wilson relation (1982), it has emerged that chiral gauge theories can be put on the lattice in a consistent way:
  - The overlap (Narayanan-Neuberger 1993,1995,1998)
  - Domain wall fermions (Kaplan-Shamir 1992, 1993, 1994)
  - Perfect action (Hasenfratz-Niedermayer 1994, 1998).

  This was believed to be impossible for a long time (Nielsen-Ninomiya, 1981, no-go theorem).

- A naive formulation of fermions on the lattice fails.

$$S_F = \frac{1}{2} \sum_x \sum_\mu \bar{\psi}(x)(\gamma_\mu \Delta_\mu + m)\psi(x) + h.c.$$

and the resulting propagator is

$$\tilde{\Delta}(k) = \frac{-i \sum_\mu \gamma_\mu \mathrm{sin} k_\mu + m}{\sum_\mu \mathrm{sin}^2 k_\mu + m^2}$$

- There is a pole for small k_μ representing the physical particle, but additional poles near $k_\mu = \pm\pi$ appears. S_F describes 16 instead of 1 particle. $\rightarrow$ Doubling problem.

- Two popular choices introduced in order to deal with this problem:
  - Wilson fermions: Get rids of the doubling species but breaks chiral symmetry explicitily by the Wilson term.
  - Staggered fermions (Kogut-Susskind): Reduce from 16 to 4 fermions and for massless fermions a chiral $U(1) \oplus U(1)$ symmetry remains.

- In the Wilson formulation the bare mass m is hidden in the hopping parameter by the relation $k = \frac{1}{8r+2m_0}$.

Wilson fermions

Propose to give up manifest SUSY on the lattice and restore it in the continuum limit.

Curci & Veneziano 1987

SUSY is broken by the lattice, by the Wilson term and a a soft breaking due to the gluino mass is present.

- SUSY is recovered in the continuum limit by tuning the bare parameters g and gluino mass $m_{\tilde{g}}$ to the SUSY point.
- The chiral and SUSY limit can be recovered simultaneously at $m_{\tilde{g}} = 0$.

Wilson fermions

The Curci and Veneziano action reads

$$S = S_G + S_F,$$

$$S_G = \frac{\beta}{2} \sum_x \sum_{\mu\nu} \left(1 - \frac{1}{N_c} \mathrm{Re}\,\mathrm{Tr}\, U_{\mu\nu}(x)\right),$$

and $\beta \equiv 2N_c/g_0^2$ correspond to the bare gauge coupling.

$$\begin{aligned} S_F \;=\; & \mathrm{Tr}\left\{ \frac{1}{2a} \Big(\bar{\lambda}(x)(\gamma_\mu - r) U_\mu^\dagger(x) \lambda(x + a\hat{\mu}) U_\mu(x) \right. \\ & -\bar{\lambda}(x + a\hat{\mu})(\gamma_\mu + r) U_\mu(x) \lambda(x) U_\mu^\dagger(x) \Big) \\ & \left. + \left(m_0 + \frac{4r}{a}\right) \bar{\lambda}(x)\lambda(x) \right\} . \end{aligned}$$

The Grassmann variables λ and $\bar{\lambda}$ are not independent

$$\bar{\lambda} = \lambda^T C\,, \qquad \lambda = C\bar{\lambda}^T\,.$$

SUSY WTi on the lattice

Another independent way to study the SUSY limit (chiral limit) is defining the SUSY WTi.

In the Wilson formulation, additional breaking terms on the lattice arise from the explicit breaking of supersymmetry.

The SUSY limit is defined to be the point in parameter space in which this breaking terms vanish and the WTi takes its continuum limit.

Nevertheless supersymmetry is not fullfilled on the lattice one might still define some SUSY transformations. One choice is (Taniguchi, hep-lat/9906026)

$$\begin{aligned}
\delta U_\mu(x) &= -agU_\mu(x)\bar{\xi}\gamma_\mu\lambda(x) - ag\bar{\xi}\gamma_\mu\lambda(x+a\widehat{\mu})U_\mu(x), \\
\delta U_\mu^\dagger(x) &= ag\bar{\xi}\gamma_\mu\lambda(x)U_\mu^\dagger(x) + agU_\mu^\dagger(x)\bar{\xi}\gamma_\mu\lambda(x+a\widehat{\mu}), \\
\delta\lambda(x) &= -\frac{i}{g}\sigma_{\rho\tau}\mathcal{G}_{\rho\tau}(x)\xi, \\
\delta\bar{\lambda}(x) &= \frac{i}{g}\bar{\xi}\sigma_{\rho\tau}\mathcal{G}_{\rho\tau}(x)
\end{aligned}$$

$\mathcal{G}_{\rho\tau}$ is the clover plaquette operator

$$\begin{aligned}
\mathcal{G}_{\rho\tau}(x) &= -\frac{1}{8a^2}\left(U_{\mu\nu}(x) - U_{\nu\mu}(x) + U_{-\mu,-\nu}(x) - U_{-\nu,-\mu}(x)\right. \\
&\quad \left.+U_{\nu,-\mu}(x) - U_{-\mu,\nu}(x) + U_{-\nu,\mu}(x) - U_{\mu,-\nu}(x)\right)
\end{aligned}$$

they reduce to the continuum SUSY transformations in the limit $a \to 0$.

SUSY WTi in Lattice Perturbation Theory

Lattice perturbation theory $\rightarrow$ we have to do gauge fixing $\rightarrow$ new terms appear in the WTi. The bare WTi is (Farchioni, A.F., Galla, Gebert, Kirchner, Montvay, Munster, Peetz, Vladikas, hep-lat/0110113 (Nucl. Phys. Proc. Suppl. 2002)

$$\langle \mathcal{O} \nabla_\mu S_\mu(x) \rangle - 2m_0 \langle \mathcal{O} \chi(x) \rangle + \left\langle \left. \frac{\delta \mathcal{O}}{\delta \bar{\xi}(x)} \right|_{\xi=0} \right\rangle - \left\langle \mathcal{O} \left. \frac{\delta S_{GF}}{\delta \bar{\xi}(x)} \right|_{\xi=0} \right\rangle - \left\langle \mathcal{O} \left. \frac{\delta S_{FP}}{\delta \bar{\xi}(x)} \right|_{\xi=0} \right\rangle = \langle \mathcal{O} X_S(x) \rangle$$

Define the local supercurrent as

$$S_\mu(x) = -\frac{2i}{g_0} \mathrm{Tr} \left\{ \mathcal{G}_{\rho\tau}(x) \sigma_{\rho\tau} \gamma_\mu \lambda(x) \right\}$$

∇_μ is the symmetric lattice derivative

$$\nabla_\mu f(x) = \frac{1}{2a} (f(x + a\hat{\mu}) - f(x - a\hat{\mu}))$$

and $\chi(x)$ corresponds to the gluino mass term

$$\chi(x) = \frac{i}{g_0} \mathrm{Tr}\left\{\mathcal{G}_{\rho\tau}(x)\sigma_{\rho\tau}\lambda(x)\right\}$$

To renormalize the lattice WTi the operator mixing should be take in consideration. Define a $\bar{X}_S(x)$ whose expectation value is forced to vanish in the continuum limit

$$X_S(x) = \bar{X}_S(x) - (Z_S - 1)\nabla_\mu S_\mu(x) - 2\tilde{m}\chi(x) - Z_T \nabla_\mu T_\mu(x)$$

Bochicchio, Maiani, Martinelli, Rossi and Testa, Nucl. Phys. B262 (1985) 331;
Donini, Guagnelli, Hernandez and Vladikas, hep-lat/9710065

$$T_\mu(x) = -\frac{2}{g} \mathrm{Tr}\left\{\mathcal{G}_{\mu\nu}(x)\gamma_\nu\lambda(x)\right\}.$$

If the operator insertion is non-gauge invariant, i.e. $\mathcal{O} := A_\alpha^a(y)\,\bar{\lambda}^b(z)$,

$$X_S(x) = \bar{X}_S(x) - (Z_S - 1)\nabla_\mu S_\mu(x) - 2\tilde{m}\chi(x) - Z_T \nabla_\mu T_\mu(x) - \sum_j Z_{B_j} B_j$$

(Farchioni, A.F., Galla, Gebert, Kirchner, Montvay, Munster, Peetz, Vladikas,
hep-lat/0110113 (Nucl. Phys. Proc. Suppl. 2002)

The B_j's denote the occurence of mixing, not only with non-gauge invariant operators (also in Taniguchi hep-lat/9906026)

$$B_1 = \frac{2}{g}\partial_\rho A_\rho \not\partial\lambda\,,\ B_2 = \frac{2}{g}A_\rho\partial_\rho \not\partial\lambda\,,\ B_3 = \frac{2}{g}\not A\partial_\rho\partial_\rho\lambda\,,$$

but also mixing with gauge invariant operators which do not vanish in the off-shell regime (but vanish in the on-shell regime)

$$B_0 = \frac{2}{g}\mathrm{Tr}\{\gamma_\rho(D_\tau\mathcal{G}_{\rho\tau}(x))\lambda(x)\}$$

Also non-Lorentz covariant terms coming from $\nabla_\mu S_\mu$, the gauge fixing term and contact terms, which appear in the off-shell regime, should also be taken in consideration.

The renormalized WTi,

$$Z_S\langle\mathcal{O}\nabla_\mu S_\mu(x)\rangle + Z_T\langle\mathcal{O}\nabla_\mu T_\mu(x)\rangle - 2(m_0-\tilde{m})Z_\chi^{-1}\langle\mathcal{O}\chi^R(x)\rangle +$$
$$Z_{CT}\langle\frac{\delta\mathcal{O}}{\delta\bar{\xi}(x)}|_{\xi=0}\rangle - Z_{GF}\langle\mathcal{O}\,\frac{\delta S_{GF}}{\delta\bar{\xi}(x)}|_{\xi=0}\rangle - Z_{FP}\langle\mathcal{O}\,\frac{\delta S_{FP}}{\delta\bar{\xi}(x)}|_{\xi=0}\rangle + \sum_j Z_{B_j}\langle\mathcal{O}B_j\rangle = 0\,.$$

A.F., hep-lat/0305020 (Phys. Rev. D70 (2004) 054504)

S_μ not only mixes with T_μ as was predicted in Curci and Veneziano, but an extra mixing with gauge variant operators and/or gauge invariant operators, which do not vanish in the off-shell regime, can appear. These extra mixing vanish by setting the renormalized gluino mass to zero and by imposing the on-shell condition on the gluino.

In order to calculate Z_T one should pick up from the WTi only those terms which contains the same Lorentz structure as S_μ and T_μ. B_i is $O(g^2)$ because on tree level does not appear.

Renormalization Constants

We are now considering

$$\mathcal{O} := A_\nu^b(y)\,\bar{\lambda}^a(z)\,,$$

In principle, each matrix element in the WTi is proportional to each element of the Γ-matrix base

$$\Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}\}\ .$$

Each matrix element can be written as

$$\langle A_\nu^b(y)\,\bar{\lambda}^a(z)\,C(x)\rangle \stackrel{FT}{\Longrightarrow} D_F(q)\cdot(C(p,q))_{amp}\cdot D_B(p)\cdot\delta_{ab}$$

The renormalization constants can be written as a power of the coupling constant g

$$Z_{operator} = Z_{operator}^{(0)} + g^2 Z_{operator}^{(2)} + \cdots\ ,$$

Also for the operators we have

$$\langle Operator\rangle = \langle Operator\rangle^{(0)} + g^2\langle Operator\rangle^{(2)} + \cdots\ ,$$

Calculate the tree-level of each operator, i.e.

$$\left(\nabla_\mu S_\mu\right)^{(0)}_{amp} \stackrel{FT}{\Longrightarrow} 2(p-q)_\mu \sigma_{\rho\nu}\gamma_\mu p_\rho \,,$$

$$\left(\nabla_\mu T_\mu\right)^{(0)}_{amp} \stackrel{FT}{\Longrightarrow} i(\not{p} p_\nu - p^2\gamma_\nu - \not{p} q_\nu + p\cdot q\gamma_\nu) \,.$$

If $p = q$, (zero momentum insertion)

$$\left(\nabla_\mu S_\mu(x)\right)^{(0)}_{amp} = \left(\nabla_\mu T_\mu(x)\right)^{(0)}_{amp} = 0.$$

These tree-levels can not be distinguished at zero momentum transfer !

General momenta p and q is required (off-shell regime)

We denote $\frac{1}{4}\mathrm{tr}(\gamma_c \mathcal{O} \nabla_\mu S_\mu)$ the projections over γ_c and $\frac{1}{4}\mathrm{tr}(\gamma_c\gamma_5 \mathcal{O} \nabla_\mu S_\mu)$ the projections over $\gamma_c\gamma_5$.

projections over the gamma matrices γ_μ and $\gamma_\mu\gamma_5$ $\rightarrow$ enough to determine Z_T.

$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\big(\nabla_\mu S_\mu\big)^{(0)}_{amp}\Big) \xrightarrow{FT} 2i(p_\alpha p_\nu - p_\alpha q_\nu - p^2\delta_{\alpha\nu} + p\cdot q\delta_{\alpha\nu})$$

$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\gamma_5\big(\nabla_\mu S_\mu\big)^{(0)}_{amp}\Big) \xrightarrow{FT} 2ip_\rho q_\sigma\varepsilon_{\nu\alpha\rho\sigma}\,.$$

$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\big(\nabla_\mu T_\mu\big)^{(0)}_{amp}\Big) \xrightarrow{FT} i(p_\alpha p_\nu - p_\alpha q_\nu - p^2\delta_{\alpha\nu} + p\cdot q\delta_{\alpha\nu})$$

$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\gamma_5\big(\nabla_\mu T_\mu\big)^{(0)}_{amp}\Big) \xrightarrow{FT} 0$$

$$-\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\big(\frac{\delta S_{GF}}{\delta\xi(x)}|_{\xi=0}\big)^{(0)}_{amp}\Big) \xrightarrow{FT} -2ip_\alpha p_\nu$$

$$-\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\gamma_5\big(\frac{\delta S_{GF}}{\delta\xi(x)}|_{\xi=0}\big)^{(0)}_{amp}\Big) \xrightarrow{FT} 0\,.$$

$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\Big(\frac{\delta\mathcal{O}}{\delta\xi(x)}|_{\xi=0}\Big)^{(0)}_{amp}\Big) \stackrel{FT}{\Longrightarrow} 2i(p_\alpha q_\nu - p\cdot q\delta_{\nu\alpha} + p^2\delta_{\alpha\nu})$$

$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\gamma_5\Big(\frac{\delta\mathcal{O}}{\delta\xi(x)}|_{\xi=0}\Big)^{(0)}_{amp}\Big) \stackrel{FT}{\Longrightarrow} -2ip_\rho q_\sigma\varepsilon_{\nu\alpha\rho\sigma}\,.$$

$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha(B_0)^{(0)}_{amp}\Big) \stackrel{FT}{\Longrightarrow} i(p_\alpha p_\nu - p^2\delta_{\alpha\nu})\,.$$

$$\frac{1}{4}\mathrm{tr}(\gamma_\alpha(B_1)_{amp}) \stackrel{FT}{\Longrightarrow} ip_\nu q_\alpha\,,$$
$$\frac{1}{4}\mathrm{tr}(\gamma_\alpha(B_2)_{amp}) \stackrel{FT}{\Longrightarrow} iq_\nu q_\alpha\,,$$
$$\frac{1}{4}\mathrm{tr}(\gamma_\alpha(B_3)_{amp}) \stackrel{FT}{\Longrightarrow} iq^2\delta_{\alpha\nu}$$

$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\gamma_5(B_{0,1,2,3})^{(0)}_{amp}\Big) \stackrel{FT}{\Longrightarrow} 0\,.$$

At tree-level we have, $Z_S^{(0)}=1, Z_T^{(0)}=0, Z_{CT}^{(0)}=1, Z_{GF}^{(0)}=1, Z_{FP}^{(0)}=0, Z_{B_i}^{(0)}=0,$

$$\langle\mathcal{O}\nabla_\mu S_\mu(x)\rangle^{(0)}+\Big\langle\frac{\delta\mathcal{O}}{\delta\bar{\xi}(x)}|_{\xi=0}\Big\rangle^{(0)}-\Big\langle\mathcal{O}\,\frac{\delta S_{GF}}{\delta\bar{\xi}(x)}|_{\xi=0}\Big\rangle^{(0)}=0\,,$$

To order g_0^2

$$\begin{aligned}&\langle\mathcal{O}\nabla_\mu S_\mu(x)\rangle^{(2)}+Z_S^{(2)}\langle\mathcal{O}\nabla_\mu S_\mu(x)\rangle^{(0)}+Z_T^{(2)}\langle\mathcal{O}\nabla_\mu T_\mu(x)\rangle^{(0)}+\\&\Big\langle\frac{\delta\mathcal{O}}{\delta\bar{\xi}(x)}|_{\xi=0}\Big\rangle^{(2)}+Z_{CT}^{(2)}\Big\langle\frac{\delta\mathcal{O}}{\delta\bar{\xi}(x)}|_{\xi=0}\Big\rangle^{(0)}-Z_{GF}^{(2)}\Big\langle\mathcal{O}\,\frac{\delta S_{GF}}{\delta\bar{\xi}(x)}|_{\xi=0}\Big\rangle^{(0)}-\\&\Big\langle\mathcal{O}\,\frac{\delta S_{GF}}{\delta\bar{\xi}(x)}|_{\xi=0}\Big\rangle^{(2)}-\sum_j Z_{B_j}^{(2)}\langle\mathcal{O}B_j(x)\rangle^{(0)}=0\,.\end{aligned}$$

Using the projections over γ_α,

$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\Big(\nabla_\mu S_\mu\Big)^{(2)}_{amp}\Big) +$$
$$Z_S^{(2)}2i(p_\alpha p_\nu - p_\alpha q_\nu - p^2\delta_{\alpha\nu} + p\cdot q\delta_{\alpha\nu}) +$$
$$Z_T^{(2)}i(p_\alpha p_\nu - p_\alpha q_\nu - p^2\delta_{\alpha\nu} + p\cdot q\delta_{\alpha\nu}) +$$
$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\Big(\frac{\delta\mathcal{O}}{\delta\bar{\xi}(x)}|_{\xi=0}\Big)^{(2)}_{amp}\Big) +$$
$$Z_{CT}^{(2)}2i(p_\alpha q_\nu - p\cdot q\delta_{\alpha\nu} + p^2\delta_{\alpha\nu}) - Z_{GF}^{(2)}2ip_\alpha p_\nu -$$
$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\Big(\frac{\delta S_{GF}}{\delta\bar{\xi}(x)}|_{\xi=0}\Big)^{(2)}_{amp}\Big) + \frac{1}{4}Z_{B_j}^{(2)}\mathrm{tr}\langle\gamma_\alpha\mathcal{O}B_j\rangle^{(0)} = 0\,,$$

and the projections over $\gamma_\alpha\gamma_5$,

$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\gamma_5\Big(\nabla_\mu S_\mu\Big)^{(2)}_{amp}\Big) + Z_S^{(2)}2ip_\rho q_\sigma\varepsilon_{\nu\alpha\rho\sigma} - Z_{CT}^{(2)}2ip_\rho q_\sigma\varepsilon_{\nu\alpha\rho\sigma} +$$
$$\frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\gamma_5\Big(\frac{\delta\mathcal{O}}{\delta\bar{\xi}(x)}|_{\xi=0}\Big)^{(2)}_{amp}\Big) - \frac{1}{4}\mathrm{tr}\Big(\gamma_\alpha\gamma_5\Big(\frac{\delta S_{GF}}{\delta\bar{\xi}(x)}|_{\xi=0}\Big)^{(2)}_{amp}\Big) +$$
$$\frac{1}{4}Z_{B_j}^{(2)}\mathrm{Tr}\langle\gamma_\alpha\gamma_5\mathcal{O}B_j\rangle^{(0)} = 0\,.$$

Our claim is that, to calculate $Z_T^{(2)}$ we can substitute

$\frac{1}{4}Z_{B_i}^{(2)}\mathrm{tr}\langle\gamma_\alpha\mathcal{O}B_i\rangle^{(0)} \to Z_{B_0}^{(2)}i(p_\alpha p_\nu - p^2\delta_{\alpha\nu}) + Z_{B_1}^{(2)}ip_\nu q_\alpha + Z_{B_2}^{(2)}iq_\nu q_\alpha + Z_{B_3}^{(2)}iq^2\delta_{\alpha\nu}$

$\frac{1}{4}Z_{B_i}^{(2)}\mathrm{Tr}\langle\gamma_\alpha\gamma_5\mathcal{O}B_j\rangle^{(0)} \to 0,$

General results

- We calculated the matrix elements of the WTi for arbitrary external momenta p and q.

- Kawai method: results of lattice integrals, two and three point form factors with massless propagators, divergent in the limit $a \to 0$.

$$I_\mu(p) = \frac{1}{a}\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin(k_\mu/2)\cos(k_\mu/2)}{\hat{k}^2 (\widehat{k+pa})^2}, p \to 0$$

$$(I_\mu(p) - \frac{\partial}{\partial p_\nu} I_\mu(p)|_{p=0}) + \frac{\partial}{\partial p_\nu} I_\mu(p)|_{p=0}$$

The first () is UV finite and can be done it in the continuum, the rest is IR divergent and should be done it on the lattice (Ellis, Martinelli, 1984).

$$\frac{1}{(16\pi)^2}\frac{p_\mu}{4}(\log(a^2p^2) + \gamma_E - F_{0000} - 2) + \frac{1}{32}p_\mu Z_0$$

- Three propagators integrals on the lattice are tabulated (Panagopoulos, Vicari, 1989) in terms of lattice numbers plus the following continuum integrals

$$\begin{aligned} I_0(p,q) &= \frac{1}{k^2(k+p)^2(k+q)^2} \\ C_\mu(p,q) &= \frac{k_\mu}{k^2(k+p)^2(k+q)^2} \\ C_{\mu\nu}(p,q) &= \frac{k_\mu\, k_\nu}{k^2(k+p)^2(k+q)^2} \\ C_{\mu\nu\rho}(p,q) &= \frac{k_\mu\, k_\nu\, k_\rho}{k^2(k+p)^2(k+q)^2} \end{aligned}$$

- With the help of ('t Hooft, Veltman 1979), (Ball, Chiu, 1980) it is possible to find out the results for I_0 and write down recursively $C_\mu(p,q)$, $C_{\mu\nu}(p,q)$, $C_{\mu\nu\rho}(p,q)$ in terms of scalar functions of $p^2, q^2, p\cdot q, I_0$ plus a Lorentz structure.

- An example:

$$C_\mu(p,q) = I_1(p,q)p_\mu + I_1(q,p)q_\mu$$

$$I_1(p,q) = \frac{1}{\Delta^2}(q^2 \log(\frac{(q-p)^2}{q^2}) - p\cdot q \log(\frac{(q-p)^2}{p^2}) + \frac{q^2 p\cdot(q-p) I_0}{2})$$

$$\Delta^2 = (p\cdot q)^2 - p^2q^2 = -p^2q^2(-\frac{(p\cdot q)^2}{p^2q^2} + 1)$$

- From the general formula (Ball-Chiu, 1980)

$$I_0 = \frac{1}{\Delta}(Li_2(\frac{p\cdot q - \Delta}{q^2}) - Li_2(\frac{p\cdot q + \Delta}{q^2}) + \frac{1}{2}\log(\frac{p\cdot q - \Delta}{p\cdot q + \Delta})\log(\frac{(q-p)^2}{q^2}))$$

- Parametrization to simplify the results:

$$\begin{aligned} \Delta &= i\,p\,q\sqrt{(1-\cos^2(\alpha))} \\ p_\mu &= |p|\,\widehat{p}_\mu \\ q_\mu &= |q|\,\widehat{q}_\mu \\ p\cdot q &= p\,q\,cos(\alpha)\ , 0 < \alpha < \pi \end{aligned}$$

- We choose: $\alpha = \frac{\pi}{2}, \rightarrow \cos(\alpha) = 0, \sin(\alpha) = 1,\ p \rightarrow q.$

- Results for small external perpendicular momenta. Projections over γ_μ and $\gamma_\mu\gamma_5$.

Define $\Delta \equiv \mathcal{O}\nabla_\mu S_\mu(x) + \frac{\delta \mathcal{O}}{\delta\bar{\xi}(x)}|_{\xi=0} - \mathcal{O}\,\frac{\delta S_{GF}}{\delta\bar{\xi}(x)}|_{\xi=0}$

$$\mathrm{tr}\langle\gamma_\alpha\Delta\rangle^{(2)} \stackrel{FT}{\Longrightarrow} A_1 q^2 \widehat{p}_\alpha \widehat{p}_\nu + A_2 q^2 \widehat{p}_\alpha \widehat{q}_\nu + (A_3 + M_3) q^2 \delta_{\alpha\nu} + M_1 q^2 \widehat{p}_\nu \widehat{q}_\alpha + M_2 q^2 \widehat{q}_\alpha \widehat{q}_\nu + P_1 q^2 \widehat{p}_\nu^2 \delta_{\nu\alpha} + P_2 q^2 \widehat{q}_\nu^2 \delta_{\nu\alpha} + \cdots$$

$$\mathrm{tr}\langle\gamma_\alpha\gamma_5\Delta\rangle^{(2)} \stackrel{FT}{\Longrightarrow} A_4 q^2 \widehat{p}_\rho \widehat{q}_\sigma \varepsilon_{\nu\alpha\rho\sigma}\,.$$

$p \cdot q\delta_{\nu\alpha}$ does not appear and $p^2\delta_{\nu\alpha}$, is mixed with $q^2\delta_{\nu\alpha}$,

$$\begin{aligned}\mathrm{tr}\langle\gamma_\alpha S_\mu\rangle^{(2)} \stackrel{FT}{\Longrightarrow}\ & N_1 q\, \widehat{p}_\mu \widehat{p}_\nu \widehat{p}_\alpha + N_2 q\, \widehat{q}_\mu \widehat{p}_\nu \widehat{p}_\alpha + \\ & N_3 q\, \widehat{p}_\mu \widehat{q}_\nu \widehat{p}_\alpha + N_4 q\, \widehat{q}_\mu \widehat{q}_\nu \widehat{p}_\alpha + N_5 q\, \widehat{p}_\mu \widehat{p}_\nu \widehat{q}_\alpha + \\ & N_6 q\, \widehat{q}_\mu \widehat{p}_\nu \widehat{q}_\alpha + N_7 q\, \widehat{p}_\mu \widehat{q}_\nu \widehat{q}_\alpha + N_8 q\, \widehat{q}_\mu \widehat{q}_\nu \widehat{q}_\alpha + \\ & Q_1 q\, \widehat{p}_\alpha \delta_{\mu\nu} + Q_2 q\, \widehat{q}_\alpha \delta_{\mu\nu} + Q_3 q\, \widehat{p}_\nu \delta_{\mu\alpha} + \\ & Q_4 q\, \widehat{q}_\nu \delta_{\mu\alpha} + Q_5 q\, \widehat{p}_\mu \delta_{\nu\alpha} + Q_6 q\, \widehat{q}_\mu \delta_{\nu\alpha} + \\ & R_1 q\, \widehat{p}_\mu \delta_{\mu\nu\alpha} + R_2 q\, \widehat{q}_\mu \delta_{\mu\nu\alpha} + \cdots\end{aligned}$$

where the coefficient A_i, M_j, Q_k, are tipically of the form

$$(C_n + C_m \mathrm{Ln}(a^2 q^2))$$

P_i, N_j, R_k, do not contain $\mathrm{Ln}(a^2q^2)$

Matching the previous eqs. we get

$$A_1 = -2iZ_S^{(2)} - iZ_T^{(2)} + 2iZ_{GF}^{(2)} - iZ_{B_0}^{(2)} ,$$
$$A_3 + M_3 = 2iZ_S^{(2)} + iZ_T^{(2)} - 2iZ_{CT}^{(2)} + iZ_{B_0}^{(2)} - iZ_{B_3}^{(2)} ,$$
$$M_1 = -iZ_{B_1}^{(2)} ,$$
$$M_2 = -iZ_{B_2}^{(2)} , \cdots$$

and

$$A_2 = 2iZ_S^{(2)} + iZ_T^{(2)} - 2iZ_{CT}^{(2)} ,$$
$$A_4 = -2iZ_S^{(2)} + 2iZ_{CT}^{(2)} .$$

The last two conditions can be explicitly solved for $Z_T^{(2)}$,

$$Z_T^{(2)} = -iA_2 - iA_4 .$$

We get

$$Z_T Z_S^{-1} \equiv Z_T^{1-loop} = 0.664 \text{ finite!}$$

To compare with numerical results

Farchioni, A.F., Galla, Gebert, Kirchner, Montvay, Munster and Vladikas, hep-lat/0111008

$$Z_T Z_S^{-1} = 0.185(7) \text{ with } \quad g^2 \to 4/2.3$$

one has to multiply by 1/2 $\to Z_T^{1-loop} = 0.332$

Exact lattice supersymmetry for the $N = 1$ Wess-Zumino model in 4 dimensions

Work based on

Marisa Bonini and A.F.:
JHEP 0409:011,2004 [hep-lat/0402034]
Phys.Rev.D71:114512,2005 [hep-lat/0504010]

Four dimensional lattice Wess-Zumino model with GW fermions

Our starting point is the paper by Fujikawa hep-lat/0205095

We show that it is actually possible to formulate the theory in such a way that the full action is invariant under a lattice superymmetry transformation at fixed lattice spacing.

The action and the transformation are written in terms of the Ginsparg-Wilson operator and reduce to their continuum expression in the limit $a \to 0$.

The lattice supersymmetry transformation is non-linear in the scalar fields and depends on the parameters m and g entering in the superpotential.

We also show that the lattice supersymmetry transformation close the algebra, a necessary ingredient to guarantee the request of supersymmetry.

The Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

implies a continuum symmetry of the fermion action which may be regarded as a lattice form of the chiral symmetry, Lüscher 1998.

The fermion lagrangian with a Yukawa interaction

$$\mathcal{L} = \bar{\psi} D\psi + g\bar{\psi}(P_+\phi\hat{P}_+ + P_-\phi^\dagger\hat{P}_-)\psi\,,$$

where

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)\,, \qquad \hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5)$$

are the lattice chiral projection operators and $\hat{\gamma}_5 = \gamma_5(1 - aD)$, is invariant under the lattice chiral transformation

$$\delta\psi = i\varepsilon\hat{\gamma}_5\psi\,, \qquad \delta\bar{\psi} = i\bar{\psi}\gamma_5\varepsilon\,, \qquad \delta\phi = -2i\varepsilon\phi\,.$$

By writing ψ in terms of two Majorana fermions

$$\psi = \chi + i\eta\,,$$

it can be seen that the interaction term couples the two Majorana fermions and therefore there is a conflict between lattice chiral symmetry and the Majorana condition Fujikawa and Ishibashi, hep-lat/0112050.

This is due to the fact that the projection operators $\hat{P}_{\pm}$ depend on D. By making the following field redefinition

$$\psi' = (1 - \frac{a}{2}D)\psi\,, \qquad \bar{\psi}' = \bar{\psi}\,,$$

the Yukawa interaction becomes

$$g\bar{\psi}'(P_{+}\phi P_{+} + P_{-}\phi^{\dagger}P_{-})\psi'$$

and the two Majorana components of ψ' decouple.

Taking advantage of this property, one can define the four dimensional Wess-Zumino on the lattice with Majorana fermions.

We start with a lagrangian defined in terms of the Ginsparg-Wilson fermions on the $d = 4$ euclidean lattice. A simple solution was given by Neuberger [1998]

$$D = \frac{1}{a}\Big(1 - X\frac{1}{\sqrt{X^\dagger X}}\Big)\,, \qquad\qquad X = 1 - aD_w\,,$$

where

$$D_w = \frac{1}{2}\gamma_\mu(\nabla^\star_\mu + \nabla_\mu) - \frac{a}{2}\nabla^\star_\mu\nabla_\mu$$

and

$$\nabla_\mu\phi(x) = \frac{1}{a}(\phi(x + a\widehat{\mu}) - \phi(x))\,, \qquad \nabla^\star_\mu\phi(x) = \frac{1}{a}(\phi(x) - \phi(x - a\widehat{\mu}))$$

It is convenient to write

$$D = D_1 + D_2$$

where

$$D_1 = \frac{1}{a}\Big(1 - \frac{1 + \frac{a^2}{2}\nabla^\star_\mu \nabla_\mu}{\sqrt{X^\dagger X}}\Big)\,, \qquad D_2 = \frac{1}{2}\gamma_\mu \frac{\nabla^\star_\mu + \nabla_\mu}{\sqrt{X^\dagger X}} \equiv \gamma_\mu D_{2\mu}\,.$$

In terms of D_1 and D_2 the Ginsparg-Wilson relation becomes

$$D_1^2 - D_2^2 = \frac{2}{a} D_1\,.$$

The action of the 4-dimensional Wess-Zumino model on the lattice

$$S_{WZ} = \sum_x \{ \frac{1}{2}\bar{\chi}(1 - \frac{a}{2}D_1)^{-1}D_2\chi - \frac{2}{a}\phi^\dagger D_1\phi + F^\dagger(1 - \frac{a}{2}D_1)^{-1}F + \frac{1}{2}m\bar{\chi}\chi$$
$$+m(F\phi + (F\phi)^\dagger) + g\bar{\chi}(P_+\phi P_+ + P_-\phi^\dagger P_-)\chi + g(F\phi^2 + (F\phi^2)^\dagger)\} ,$$

where ϕ and F are scalar fields and χ is a Majorana fermion which satisfies the Majorana condition

$$\bar{\chi} = \chi^T C$$

and C is the charge conjugation matrix which satisfies

$$C^T = -C \,, \qquad CC^\dagger = 1 \,.$$

Moreover, our conventions are

$$C\gamma_\mu C^{-1} = -(\gamma_\mu)^T$$
$$C\gamma_5 C^{-1} = (\gamma_5)^T \,.$$

In the continuum limit reduces to the continuum Wess-Zumino action

$$
\begin{aligned}
S \;=\; & \int \{ \frac{1}{2}\bar{\chi}(\partial\!\!\!/ + m)\chi + \phi^{\dagger}\partial^{2}\phi + F^{\dagger}F + m(F\phi + (F\phi)^{\dagger}) \\
& + g\bar{\chi}(P_{+}\phi P_{+} + P_{-}\phi^{\dagger}P_{-})\chi + g(F\phi^{2} + (F\phi^{2})^{\dagger}) \Big\} \, .
\end{aligned}
$$

The supersymmetric transformation

If one defines the real components by

$$\phi \to \frac{1}{\sqrt{2}}(A + iB)\,, \quad F \to \frac{1}{\sqrt{2}}(F - iG)$$

the WZ model $S_{WZ} = S_0 + S_{int}$

$$S_0 = \sum_x \{\frac{1}{2}\bar{\chi}(1 - \frac{a}{2}D_1)^{-1}D_2\chi - \frac{1}{a}(AD_1A + BD_1B) + \frac{1}{2}F(1 - \frac{a}{2}D_1)^{-1}F + \frac{1}{2}G(1 - \frac{a}{2}D_1)^{-1}G\}\,,$$

$$S_{int} = \sum_x \{\frac{1}{2}m\bar{\chi}\chi + m(FA + GB) + \frac{1}{\sqrt{2}}g\bar{\chi}(A + i\gamma_5 B)\chi + \frac{1}{\sqrt{2}}g[F(A^2 - B^2) + 2G(AB)]\}\,.$$

S_0 (free part) is invariant under the lattice supersymmetry transformation [Fujikawa 2002]

$$\begin{aligned}
\delta A &= \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
\delta B &= -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
\delta\chi &= -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon \\
\delta F &= \bar{\varepsilon}D_2\chi \\
\delta G &= i\bar{\varepsilon}D_2\gamma_5\chi \,.
\end{aligned}$$

In fact, the variation of S_0 under the this transformation is

$$\begin{aligned}
\delta S_0 &= \\
&= \sum_x \{\bar{\chi}(1 - \frac{a}{2}D_1)^{-1}D_2[-D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon] - \frac{2}{a}\bar{\chi}\varepsilon D_1 A \\
&\qquad + \frac{2i}{a}\bar{\chi}\gamma_5\varepsilon D_1 B + (\bar{\varepsilon}D_2\chi)(1 - \frac{a}{2}D_1)^{-1}F + i(\bar{\varepsilon}D_2\gamma_5\chi)(1 - \frac{a}{2}D_1)^{-1}G\} \,.
\end{aligned}$$

and integrating by part *, this variation becomes

$$\sum_x \{ -\bar{\chi}\varepsilon\big[(1-\frac{a}{2}D_1)^{-1}D_2^2+\frac{2}{a}D_1\big]A+i\bar{\chi}\gamma_5\varepsilon\big[(1-\frac{a}{2}D_1)^{-1}D_2^2+\frac{2}{a}D_1\big]B$$
$$-\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2(F-i\gamma_5 G)\varepsilon+\bar{\chi}D_2\varepsilon(1-\frac{a}{2}D_1)^{-1}F+i\bar{\chi}D_2\gamma_5\varepsilon(1-\frac{a}{2}D_1)^{-1}G\}$$
$$=0\,,$$

where we used the Ginsparg-Wilson relation, which implies

$$(1-\frac{a}{2}D_1)^{-1}D_2^2=-\frac{2}{a}D_1\,.$$

*For instance, for any scalar function $\mathcal{F}$ one has $\mathcal{F}\bar{\varepsilon}D_2\chi=\bar{\chi}D_2\mathcal{F}\varepsilon$.

Failure of the Leibniz rule

The variation of S_{int} under the susy transformation does not vanish because of the failure of the Leibniz rule at finite lattice spacing [Dondi and Nicolai 1977, Fujikawa 2002]

$$\frac{1}{a}(f(x+a)g(x+a) - f(x)g(x)) =$$
$$= \frac{1}{a}(f(x+a) - f(x))g(x) + \frac{1}{a}f(x)(g(x+a) - g(x))$$
$$+ a\frac{1}{a}(f(x+a) - f(x))\frac{1}{a}(g(x+a) - g(x))$$
$$= (\nabla f(x))g(x) + f(x)(\nabla g(x)) + a(\nabla f(x))(\nabla g(x))$$

the breaking of supersymmetry is of order $O(a)$.

- In order to discuss the symmetry properties of the lattice Wess-Zumino model one possibility is to modify the action by adding irrelevant terms which make invariant the full action.

- Alternatively, one can modify the supersymmetry transformation in such a way that the action has an exact symmetry for $a \neq 0$.

Since the transformation leaves invariant the free part of the action, this modification must vanish for $g = 0$.

$$
\begin{aligned}
&\delta A = \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
&\delta B = -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
&\delta\chi = -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon + gR\varepsilon \\
&\delta F = \bar{\varepsilon}D_2\chi \\
&\delta G = i\bar{\varepsilon}D_2\gamma_5\chi
\end{aligned}
$$

- R to be determined by requiring that the variation of the action vanishes.
  - We assume that R depends on the scalar fields and their derivatives and not on χ.

Exact susy transformation for the full action

The variation of the Wess-Zumino action under the transformation is

$$\begin{aligned}\delta S_{WZ} &= \sum_x \{g\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2R\varepsilon - m\bar{\chi}[D_2(A-i\gamma_5 B)\varepsilon + (F-i\gamma_5 G)\varepsilon - gR\varepsilon] \\ &\quad + m(A\bar{\varepsilon}D_2\chi + F\bar{\chi}\varepsilon + iB\bar{\varepsilon}D_2\gamma_5\chi - iG\bar{\chi}\gamma_5\varepsilon) + \frac{g}{\sqrt{2}}\bar{\chi}(\bar{\varepsilon}\chi + \gamma_5(\bar{\varepsilon}\gamma_5\chi))\chi \\ &\quad -\sqrt{2}g\bar{\chi}(A+i\gamma_5 B)[D_2(A-i\gamma_5 B)\varepsilon + (F-i\gamma_5 G)\varepsilon - gR\varepsilon] \\ &\quad + \frac{g}{\sqrt{2}}[(A^2-B^2)\bar{\varepsilon}D_2\chi + 2FA\bar{\chi}\varepsilon + 2iFB\bar{\chi}\gamma_5\varepsilon \\ &\quad + 2iAB\bar{\varepsilon}D_2\gamma_5\chi + 2GB\bar{\chi}\varepsilon - 2iGA(\bar{\chi}\gamma_5\varepsilon)]\}\,.\end{aligned}$$

By using the Fierz identity, terms with four fermions cancel as in the continuum.

Moreover, g independent terms cancel out after an integration by part, and one is left with

$$\begin{aligned}\delta S_{WZ} &= \sum_x \{g\bar{\chi}[(1-\frac{a}{2}D_1)^{-1}D_2R + mR]\varepsilon - \frac{g}{\sqrt{2}}[2\bar{\chi}(A+i\gamma_5 B)D_2(A-i\gamma_5 B)\varepsilon \\ &\qquad - \bar{\chi}D_2(A-i\gamma_5 B)^2\varepsilon] + \sqrt{2}g^2\bar{\chi}(A+i\gamma_5 B)R\varepsilon\}\,.\end{aligned}$$

The function R is determined by imposing the vanishing of δS_{WZ} order by order in g. By expanding R in powers of g

$$R = R^{(1)} + gR^{(2)} + \cdots$$

and imposing the symmetry condition order by order in perturbation theory, we find

$$R^{(1)} = ((1 - \frac{a}{2}D_1)^{-1}D_2 + m)^{-1}\Delta L$$

with

$$\begin{aligned}\Delta L &\equiv \frac{1}{\sqrt{2}}(2(A + i\gamma_5 B)D_2(A - i\gamma_5 B) - D_2(A - i\gamma_5 B)^2) \\ &= \frac{1}{\sqrt{2}}\{2(AD_2A - BD_2B) - D_2(A^2 - B^2) \\ &\quad + 2i\gamma_5[(AD_2B + BD_2A) - D_2(AB)]\}\,.\end{aligned}$$

To order g^2 one has

$$R^{(2)} = -\sqrt{2}((1-\frac{a}{2}D_1)^{-1}D_2+m)^{-1}(A+i\gamma_5 B)((1-\frac{a}{2}D_1)^{-1}D_2+m)^{-1}\Delta L\,,$$

and for $n \geq 2$

$$R^{(n)} = -\sqrt{2}((1-\frac{a}{2}D_1)^{-1}D_2+m)^{-1}(A+i\gamma_5 B)R^{(n-1)}\,.$$

The formal solution is

$$[(1-\frac{a}{2}D_1)^{-1}D_2+m+\sqrt{2}g(A+i\gamma_5 B)]R = \Delta L\,.$$

- $R \to 0$ for $a \to 0$, since ΔL vanishes in this limit.
- ΔL is different from zero because of the breaking of the Leibniz rule for a finite lattice spacing.

The algebra

By the commutator of two supersymmetries one find a transformation which is still a symmetry of the Wess-Zumino action, i.e. the transformations of the fields form a closed algebra, order by order in g.

Up to order g^1, (the rest can be generalized!)

Two supersymmetry transformations on the scalar field A give

$$\begin{aligned}\delta_1\delta_2 A &= \delta_1(\bar\varepsilon_2\chi)\\ &= -\bar\varepsilon_2[D_2(A - i\gamma_5 B)\varepsilon_1 + (F - i\gamma_5 G)\varepsilon_1 - gR\varepsilon_1]\end{aligned}$$

and their commutator yields

$$[\delta_2, \delta_1]A = -2\bar\varepsilon_1 D_2 \varepsilon_2 A + g(\bar\varepsilon_1 R\varepsilon_2 - \bar\varepsilon_2 R\varepsilon_1)\,.$$

The order g^1 of the second term on the r.h.s. reads

$$g(\bar\varepsilon_1 R^{(1)}\varepsilon_2 - \bar\varepsilon_2 R^{(1)}\varepsilon_1) =$$
$$\sqrt{2}g\bar\varepsilon_2 \frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_2(A^2-B^2) - 2(AD_2A - BD_2B)]\varepsilon_1$$

Finally, the commutator of two supersymmetries on the scalar field A is

$$[\delta_2,\delta_1]A = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}A + \frac{g}{\sqrt{2}}\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(A^2-B^2)-2(AD_{2\mu}A-BD_{2\mu}B)]\}\,.$$

Similarly, the commutators of two supersymmetries on the other fields, up to terms of order g^1, are

$$[\delta_2,\delta_1]B = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}B + \sqrt{2}g\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(AB)-(AD_{2\mu}B+BD_{2\mu}A)]\}\,,$$

$$[\delta_2,\delta_1]F = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}F - \frac{g}{\sqrt{2}}\frac{D_2^2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(A^2-B^2)-2(AD_{2\mu}A-BD_{2\mu}B)]\}\,,$$

$$[\delta_2,\delta_1]G = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}G - \sqrt{2}g\frac{D_2^2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(AB)-(AD_{2\mu}B+BD_{2\mu}A)]\}$$

and

$$
\begin{aligned}
[\delta_2, \delta_1]\chi = & -2\bar{\varepsilon}_1 \gamma_\mu \varepsilon_2 \{ D_{2\mu}\chi \\
& - \frac{g}{\sqrt{2}} \frac{m(1 - \frac{a}{2}D_1) - D_2}{m^2(1 - \frac{a}{2}D_1) + \frac{2}{a}D_1} \big(D_2(A - i\gamma_5 B)\gamma_\mu \chi + (A + i\gamma_5 B) D_2 \gamma_\mu \chi \\
& \qquad - D_2[(A - i\gamma_5 B)\gamma_\mu \chi] \big) \} \, .
\end{aligned}
$$

Therefore, the general expression of these commutators is

$$[\delta_1, \delta_2]\Phi = \alpha^\mu P_\mu^\Phi(\Phi)\,, \qquad \Phi = (A, B, F, G, \chi)\,,$$

where $\alpha^\mu = -2\bar{\varepsilon}_2\gamma^\mu\varepsilon_2$ and $P_\mu^\Phi(\Phi)$ are polynomials in Φ defined as

$$P_\mu^\Phi(\Phi) = D_{2\mu}\Phi + O(g)$$

We have verified that the closure works, i.e. the action is invariant under the transformation (up to terms of order g^1).

$$\Phi \to \Phi + \alpha^\mu P_\mu^\Phi(\Phi)$$

Notice that, in the continuum limit $D_{2\mu} \to \partial_\mu$ and the transformation reduces to

$$\Phi \to \Phi + \alpha^\mu \partial_\mu \Phi$$

Ward-Takahashi identity

The WTi is derived from the generating functional

$$Z[\Phi, J] = \int \mathcal{D}\Phi \, \exp -(S_{WZ} + S_J)$$

where S_J is the source term

$$S_J = \sum_x J_\Phi \cdot \Phi \equiv \sum_x \left\{ J_A \, A + J_B \, B + J_F \, F + J_G \, G + \bar{\eta}\chi \right\} .$$

Using the invariance of both the Wess-Zumino action and the measure with respect to the lattice supersymmetry transformation, the WTi reads

$$\langle J_\Phi \cdot \delta\Phi \rangle_J = 0 \, .$$

The propagators for the scalar and fermion fields are

$$\begin{aligned}
\langle AA \rangle &= \langle BB \rangle = -\mathcal{M}^{-1}(1-\frac{a}{2}D_1)^{-1} \\
\langle FF \rangle &= \langle GG \rangle = \frac{2}{a}\mathcal{M}^{-1}D_1 = -\mathcal{M}^{-1}(1-\frac{a}{2}D_1)^{-1}D_2^2 \\
\langle AF \rangle &= \langle BG \rangle = m\,\mathcal{M}^{-1} \\
\langle \chi\bar{\chi} \rangle &= ((1-\frac{a}{2}D_1)^{-1}D_2 + m)^{-1} = -\mathcal{M}^{-1}((1-\frac{a}{2}D_1)^{-1}D_2 - m)\,,
\end{aligned}$$

where

$$\mathcal{M} = [\frac{2}{a}D_1(1-\frac{a}{2}D_1)^{-1} + m^2] \qquad (1)$$

and the Ginsparg-Wilson relation has been used to rewrite the auxiliary fields propagators.

One-point Ward-Takahashi identity

The simplest WTi which is obtained by taking the derivative with respect to $\bar{\eta}$ and setting to zero all the sources

$$\langle D_2(A - i\gamma_5 B)\rangle + \langle F\rangle - i\gamma_5\langle G\rangle - g\langle R\rangle = 0\,.$$

The order $O(g)$ is

$$\langle D_2(A - i\gamma_5 B)\rangle^{(1)} + \langle F\rangle^{(1)} - i\gamma_5\langle G\rangle^{(1)} - g\langle R^{(1)}\rangle^{(0)} = 0\,,$$

$\langle \mathcal{O}\rangle^{(n)}$: $n-$order in g of $\mathcal{O}$.

By substituting the propagators one can show that all the terms of this WTi are zero.

$$\langle D_2 A\rangle^{(1)} \sim D_{2xy}\Big[\langle A_y F_u\rangle[\langle A_u A_u\rangle - \langle B_u B_u\rangle] + \langle A_y A_u\rangle[2\langle A_u F_u\rangle + 2\langle B_u G_u\rangle + \mathrm{Tr}\langle\bar{\chi}_u \chi_u\rangle]\Big] = 0$$

and similarly

$$\langle F_x\rangle^{(1)} \sim \langle F_x F_u\rangle[\langle A_u A_u\rangle - \langle B_u B_u\rangle] + \langle F_x A_u\rangle[2\langle A_u F_u\rangle + 2\langle B_u G_u\rangle + \mathrm{Tr}\langle\bar{\chi}_u \chi_u\rangle] = 0\,.$$

The Feynman diagrams of $\langle F_x\rangle^{(1)}$ are

The vanishing of the A and F one-point functions is due to the exact cancellation of the tadpole diagrams on the lattice.

Similarly, the G and B one-point functions are zero at this order due to the matrix γ_5 inserted in the fermion loop.

Finally also the contribution with R vanishes,

$$\langle R^{(1)}\rangle^{(0)} = ((1-\frac{a}{2}D_1)^{-1}D_2 + m)^{-1}\langle\Delta L_y\rangle^{(0)}$$
$$= \langle\chi_x\bar{\chi}_y\rangle\big[2\langle A_y D_{2yz} A_z\rangle - 2\langle B_y D_{2yz} B_z\rangle - D_{2yz}\langle A_z A_z\rangle + D_{2yz}\langle B_z B_z\rangle\big] = 0\,,$$

Two-point Ward-Takahashi identity

A more interesting WTi that relates the fermion and scalar two-point functions.

Taking the derivative of $\langle J_\Phi \cdot \delta\Phi \rangle_J = 0$ with respect to $\bar{\eta}$ and J_A and setting to zero all the sources one obtains

$$\langle \chi_y \bar{\chi}_x \rangle - \langle D_{2yz}(A_z - i\gamma_5 B_z) A_x \rangle - \langle (F_y - i\gamma_5 G_y) A_x \rangle + g \langle R_y A_x \rangle = 0\,.$$

This identity is trivially satisfied at tree level if one uses the corresponding propagators.

The next non-trivial order is g^2 which corresponds to the one-loop diagrams and can be written as

$$\langle\chi_y\bar{\chi}_x\rangle^{(2)} - \langle D_{2yz}(A_z - i\gamma_5 B_z)A_x\rangle^{(2)} - \langle(F_y - i\gamma_5 G_y)A_x\rangle^{(2)} + g\left(\langle R_y^{(1)}A_x\rangle^{(1)} + g\langle R_y^{(2)}A_x\rangle^{(0)}\right) = 0\,,$$

Applying the Wick expansion the first term reads

$$\langle\chi_y\bar{\chi}_x\rangle^{(2)} = \frac{g^2}{4}\langle\chi_y\bar{\chi}_x\sum_{zu}[\bar{\chi}(A + i\gamma_5 B)\chi + F(A^2 - B^2) + 2GAB]_z \times [\bar{\chi}(A + i\gamma_5 B)\chi + F(A^2 - B^2) + 2GAB]_u\rangle^{(0)}\,.$$

Let us isolate among the various contributions the tadpole ones

$$\begin{aligned}\langle\chi_y\bar{\chi}_x\rangle_T^{(2)} = g^2\sum_{zu}\{&\langle\chi_y\bar{\chi}_z\rangle\langle\chi_z\bar{\chi}_x\rangle[\langle A_z F_u\rangle(\langle A_u A_u\rangle - \langle B_u B_u\rangle)\\ &+ 2\langle A_z A_u\rangle(\langle A_u F_u\rangle + \langle B_u G_u\rangle) - \langle A_z A_u\rangle \mathrm{Tr}\langle\chi_u\bar{\chi}_u\rangle]\\ &+ \langle\chi_y\bar{\chi}_z\rangle\gamma_5\langle\chi_z\bar{\chi}_x\rangle\langle B_z B_u\rangle \mathrm{Tr}\langle\chi_u\gamma_5\bar{\chi}_u\rangle\}\,.\end{aligned}$$

Using the relations $\mathrm{Tr}\langle\chi\gamma_5\bar{\chi}\rangle = 0$ and $\mathrm{Tr}\langle\chi\bar{\chi}\rangle = 4\langle AF\rangle = 4\langle GB\rangle$ one can demonstrate that the tadpole contributions cancel out.

This property is general and also holds for the other terms of the WTi.

Therefore, one is left with the connected non tadpoles diagrams

$$\langle\chi_y\bar{\chi}_x\rangle^{(2)}_{NT} = 2g^2\sum_{uz}\{\langle\chi_y\bar{\chi}_z\rangle\langle\chi_z\bar{\chi}_u\rangle\langle\chi_u\bar{\chi}_x\rangle\langle A_zA_u\rangle - \langle\chi_y\bar{\chi}_z\rangle\gamma_5\langle\chi_z\bar{\chi}_u\rangle\gamma_5\langle\chi_u\bar{\chi}_x\rangle\langle B_zB_u\rangle\}\,.$$

The corresponding Feynman diagrams are

The non-tadpole contributions to the second term of the WTi are (sum over repeated indices z, u, w is understood)

$$\begin{aligned}\langle D_{2yz}(A_z - i\gamma_5 B_z)A_x\rangle^{(2)}_{NT} = g^2\{ & D_{2yz}\langle A_z A_u\rangle[\mathrm{Tr}(\langle\chi_u\bar{\chi}_w\rangle\langle\chi_w\bar{\chi}_u\rangle) + 2\langle A_u A_w\rangle\langle F_u F_w\rangle \\ & + 2\langle B_u B_w\rangle\langle G_u G_w\rangle + 2\langle F_u A_w\rangle\langle A_u F_w\rangle + 2\langle B_u G_w\rangle\langle G_u B_w\rangle]\langle A_w A_x\rangle \\ & + D_{2yz}\langle A_z F_u\rangle[\langle A_u A_w\rangle\langle A_u A_w\rangle + \langle B_u B_w\rangle\langle B_u B_w\rangle]\langle F_w A_x\rangle \\ & + 2D_{2yz}\langle A_z F_u\rangle[\langle A_u A_w\rangle\langle A_u F_w\rangle - \langle B_u B_w\rangle\langle B_u G_w\rangle]\langle A_w A_x\rangle \\ & + 2D_{2yz}\langle A_z A_u\rangle[\langle A_u A_w\rangle\langle F_u A_w\rangle - \langle B_u B_w\rangle\langle G_u B_w\rangle]\langle F_w A_x\rangle\}\,.\end{aligned}$$

The corresponding Feynman diagrams are

The non-tadpole contributions to the third term of WTi are

$$
\begin{aligned}
\langle (F_y - i\gamma_5 G_y) A_x \rangle^{(2)}_{NT} = g^2 \{ & 2\langle F_y A_u \rangle [\frac{1}{2} \mathrm{Tr}(\langle \chi_u \bar{\chi}_w \rangle \langle \chi_w \bar{\chi}_u \rangle) + \langle F_u F_w \rangle \langle A_u A_w \rangle \\
& + \langle F_u A_w \rangle \langle A_u F_w \rangle + \langle G_u G_w \rangle \langle B_u B_w \rangle + \langle B_u G_w \rangle \langle G_u B_w \rangle] \langle A_w A_x \rangle \\
& + \langle F_y F_u \rangle [\langle A_u A_w \rangle \langle A_u A_w \rangle + \langle B_u B_w \rangle \langle B_u B_w \rangle] \langle F_w A_x \rangle \\
& + 2\langle F_y A_u \rangle [\langle F_u A_w \rangle \langle A_u A_w \rangle - \langle G_u B_w \rangle \langle B_u B_w \rangle] \langle F_w A_x \rangle \\
& + 2\langle F_y F_u \rangle [\langle A_u A_w \rangle \langle A_u F_w \rangle - \langle B_u B_w \rangle \langle B_u G_w \rangle] \langle A_w A_x \rangle \\
& - \gamma_5 \langle G_y B_w \rangle \mathrm{Tr}(\gamma_5 \langle \bar{\chi}_w \chi_u \rangle \langle \bar{\chi}_u \chi_w \rangle) \langle A_u A_x \rangle \}
\end{aligned}
$$

The corresponding Feynman diagrams are

For the terms of the WTi involving R one finds

$$\langle R_y^{(1)} A_x \rangle^{(1)} = -\frac{g}{\sqrt{2}} \langle \chi \bar{\chi} \rangle_{yz} \langle \Delta L_z [\bar{\chi}(A + i\gamma_5 B)\chi + F(A^2 - B^2) + 2GAB]_u A_x \rangle^{(0)} ,$$

Also in this case the tadpole diagrams cancel out and one is left with

$$\begin{aligned}\langle R_y^{(1)} A_x \rangle_{NT}^{(1)} &= -g \langle \chi \bar{\chi} \rangle_{yz} \\ &\times \{2[\langle A_z F_w \rangle D_{2zu} \langle A_u A_w \rangle + \langle A_z A_w \rangle D_{2zu} \langle A_u F_w \rangle - D_{2zu} \langle A_u F_w \rangle \langle A_u A_w \rangle \\ &- \langle B_z G_w \rangle D_{2zu} \langle B_u B_w \rangle - \langle B_z B_w \rangle D_{2zu} \langle B_u G_w \rangle + D_{2zu} \langle B_u B_w \rangle \langle B_u G_w \rangle] \langle A_w A_x \rangle \\ &+ [2 \langle A_z A_w \rangle D_{2zu} \langle A_u A_w \rangle - D_{2zu} \langle A_u A_w \rangle \langle A_u A_w \rangle \\ &\quad + 2 \langle B_z B_w \rangle D_{2zu} \langle B_u B_w \rangle - D_{2zu} \langle B_u B_w \rangle \langle B_u B_w \rangle] \langle F_w A_x \rangle \} .\end{aligned}$$

The corresponding Feynman diagrams are

And the last term of WTi is

$$
\begin{aligned}
\langle R_y^{(2)} A_x \rangle^{(0)} &= -\sqrt{2} \langle \chi \bar{\chi} \rangle_{yz} \langle (A_z + i\gamma_5 B_z) \langle \chi \bar{\chi} \rangle_{zw} \Delta L_w A_x \rangle^{(0)} \\
&= -2\{ \langle \chi_y \bar{\chi}_z \rangle \langle \chi_z \bar{\chi}_w \rangle [\langle A_z A_w \rangle D_{2wu} \langle A_u A_x \rangle + \langle A_w A_x \rangle D_{2wu} \langle A_z A_u \rangle \\
&\qquad - D_{2wu} \langle A_z A_u \rangle \langle A_u A_x \rangle] \\
&\quad - \langle \chi_y \bar{\chi}_z \rangle \gamma_5 \langle \chi_z \bar{\chi}_w \rangle \gamma_5 [\langle B_z B_w \rangle D_{2wu} \langle A_u A_x \rangle + \langle A_w A_x \rangle D_{2wu} \langle B_z B_u \rangle \\
&\qquad - D_{2wu} \langle B_z B_u \rangle \langle A_u A_x \rangle]\} \,,
\end{aligned}
$$

and the corresponding Feynman diagrams are

Calculation in momentum space

Collecting all the differents diagrams of

$$\langle \chi_y \bar{\chi}_x \rangle - \langle D_{2yz}(A_z - i\gamma_5 B_z) A_x \rangle - \langle (F_y - i\gamma_5 G_y) A_x \rangle + g\langle R_y A_x \rangle = 0\,.$$

For the fermion two-point function,

$$\langle \chi(p)\bar{\chi}(q)\rangle^{(2)} = 4g^2(2\pi)^4\delta^4(p+q)\big(D_2(p) - m(1-\frac{a}{2}D_1(p))\big)\int_k \mathcal{G}^{-1}(p,k)D_2(p+k) \times \big(D_2(p) - m(1-\frac{a}{2}D_1(p))\big)\,,$$

where

$$\mathcal{G}(p,k) = [\mathcal{M}(p)((1-\frac{a}{2}D_1(p))]^2[\mathcal{M}(k)((1-\frac{a}{2}D_1(k))][\mathcal{M}(k+p)((1-\frac{a}{2}D_1(k+p))]$$

and $D_1(p)$, $D_2(p)$ and $\mathcal{M}(p)$ are the Fourier transform of the operators given before.

$$\langle D_2(p)(A(p) - i\gamma_5 B(p))A(q)\rangle^{(2)} = \langle D_2(p)A(p)A(q)\rangle^{(2)} = g^2(2\pi)^4\delta^4(p+q)D_2(p)$$
$$\times \int_k \mathcal{G}^{-1}(p,k)[2m^2(1 - \frac{a}{2}D_1(p))^2 - \mathrm{Tr}[D_2(k)D_2(p+k)] + 4D_2^2(k)]$$

and

$$\langle (F(p) - i\gamma_5 G(p))A(q)\rangle^{(2)} = \langle F(p)A(q)\rangle^{(2)} = mg^2(2\pi)^4\delta^4(p+q)(1 - \frac{a}{2}D_1(p))$$
$$\times \int_k \mathcal{G}^{-1}(p,k)[\mathrm{Tr}\big(D_2(k)D_2(p+k)\big) - 4D_2^2(k) - 2D_2^2(p)]\,,$$

respectively.

$$\langle R^{(1)}(p)A(q)\rangle^{(1)} = 2mg^2(2\pi)^4\delta^4(p+q)(1-\frac{a}{2}D_1(p))\big(D_2(p)-m(1-\frac{a}{2}D_1(p))\big)$$
$$\times\int_k \mathcal{G}^{-1}(p,k)\big(2D_2(p+k)-D_2(p)\big)$$

and

$$\langle R^{(2)}(p)A(q)\rangle^{(0)} = -4g^2(2\pi)^4\delta^4(p+q)\big(D_2(p)-m(1-\frac{a}{2}D_1(p))\big)$$
$$\times\int_k \mathcal{G}^{-1}(p,k)D_2(p+k)\big(D_2(k)+D_2(p)-D_2(p+k)\big)\,.$$

In order to verify that the WTi is exactly satisfied it is convenient to arrange the various terms according to the powers of m.

Setting $m = 0$ one has

$$4g^2(2\pi)^4\delta^4(p+q)\int_k \mathcal{G}^{-1}(p,k)[D_2(p)D_2(p+k)D_2(p) + D_2(p)(D_2(k)\cdot D_2(p+k) - D_2^2(k)) - D_2(p)D_2(p+k)(D_2(k)+D_2(p)-D_2(p+k))]\,,$$

where $\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 4\delta_{\mu\nu}$ has been used.

Taking advantage of the invariance of $\mathcal{G}(p,k)$ under the change of variables $k \to -k-p$, one can replace $D_2(p+k)D_2(k)$ with $\frac{1}{2}\{D_2(p+k), D_2(k)\} = D_2(p+k)\cdot D_2(k)$ and therefore the integrand exactly vanishes.

The terms proportional to m add up to

$$g^2(2\pi)^4\delta^4(p+q)(1-\frac{a}{2}D_1(p))\int_k \mathcal{G}^{-1}(p,k)[-4(D_2(p)D_2(p+k)+D_2(p+k)D_2(p)) \\ +4D_2^2(k)+2D_2^2(p)-4D_2(k)\cdot D_2(p+k)+D_2(p)(4D_2(p+k)-2D_2(p)) \\ +4D_2(p+k)(D_2(k)+D_2(p)-D_2(p+k))]\,.$$

Performing the substitution $D_2(p+k)D_2(k) \to D_2(p+k)\cdot D_2(k)$ this expression vanishes.

Finally, the contribution left is the one proportional to m^2,

$$g^2(2\pi)^4\delta^4(p+q)(1-\frac{a}{2}D_1(p))^2\int_k \mathcal{G}^{-1}(p,k)[4D_2(p+k)-2D_2(p) - 2(2D_2(p+k)-D_2(p))]$$

which is trivially zero.

This end up our proof that the WTi is exactly satisfied at finite lattice spacing.

Continuum limit

In this section we study the continuum limit of the WTi and discuss the restoration of the continuum supersymmetry in this limit.

This will clarify the mechanism of cancellation between the different terms in the WTi and the role of the operator $\langle R(p)A(q)\rangle^{(2)}$.

Using the standard notation for the operator D in one has
(Kikukawa and Yamada, Phys. Lett. B448 (1999) 265)

$$D(p) = [\frac{-i\sum_\mu \gamma_\mu \sin(p_\mu a)}{2[\omega(p)+b(p)]} + \frac{a}{2}]^{-1}$$

where

$$\omega(p) = \frac{1}{a}\Big[\sum_\mu \sin^2(p_\mu a) + (ab(p))^2\Big]^{1/2}$$

and

$$b(p) = \frac{1}{a}\Big[\sum_\mu 2\sin^2(\frac{p_\mu a}{2}) - 1\Big]\,.$$

With this notation, the operators D_1 and D_2 $(D = D_1 + D_2)$ are

$$D_1(p) = \frac{\omega(p)+b(p)}{a\omega(p)}$$

and

$$D_2(p) = \frac{i}{a^2\omega(p)}\sum_\mu \gamma_\mu \sin(p_\mu a)\,.$$

Similarly,

$$(1 - \frac{a}{2}D_1) = \frac{\omega - b}{2\omega}$$

and

$$\mathcal{M}^{-1}(1 - \frac{a}{2}D_1)^{-1} = 2\omega a^2\big[4(\omega + b) + a^2m^2(\omega - b)\big]^{-1}.$$

Each term in the WTi is a function of the external momenta p and can be written as

$$I(p) = \int \frac{d^4k}{(2\pi)^4} F(k,p)$$

where the integration momenta $k \in [-\frac{\pi}{a}, \frac{\pi}{a}]$.

If the integral $I(p)$ is ultraviolet convergent its continuum limit is obtained substituting the function $F(k,p)$ with its continuum equivalent. Otherwise, if $I(p)$ is divergent and contains only massive propagators so that $F(k,p)$ is finite for any set of exceptional momenta, one can use the lattice version of the BPHZ technique by writing

$$I(p) \equiv I^c(p) + I^l(p)\,,$$

where

$$I^c(p) = \int \frac{d^4k}{(2\pi)^4} \Big[F(k,p) - \sum_{n=0}^{n_F} \frac{1}{n!} p_{\mu_1} \ldots p_{\mu_n} \big(\frac{\partial}{\partial p_{\mu_1}} \cdots \frac{\partial}{\partial p_{\mu_n}} F(k,p)\big)_{p=0}\Big]\,,$$

$$I^l(p) = \int \frac{d^4k}{(2\pi)^4} \sum_{n=0}^{n_F} \frac{1}{n!} p_{\mu_1} \ldots p_{\mu_n} \big(\frac{\partial}{\partial p_{\mu_1}} \cdots \frac{\partial}{\partial p_{\mu_n}} F(k,p)\big)_{p=0}$$

For the fermion two point function one has to consider the following integral

$$-\frac{i}{a}\int\frac{d^4k}{(2\pi)^4}\frac{1}{[(\omega'+b')_k+\frac{a^2m^2}{4}(\omega'-b')_k]}\frac{\omega'_{p+k}}{[(\omega'+b')_{p+k}+\frac{a^2m^2}{4}(\omega'-b')_{p+k}]}\sum_\mu \gamma_\mu \sin(k_\mu)$$

where k has been rescaled to $k \to k/a$ and we have defined

$$\omega'_k \equiv a\omega(k/a) = \left[1-4\sum_\mu \sin^4(\frac{k_\mu}{2})+4\Big(\sum_\mu \sin^2(\frac{k_\mu}{2})\Big)^2\right]^{1/2}$$

and

$$\omega'_{p+k} \equiv a\omega(p+k/a) = \left[1-4\sum_\mu \sin^4(\frac{(k+ap)_\mu}{2})+4\Big(\sum_\mu \sin^2(\frac{(k+ap)_\mu}{2})\Big)^2\right]^{1/2}.$$

Similarly, $b'_k \equiv ab(k/a)$ and $b'_{p+k} \equiv ab(p+k/a)$.

The factor $1/a$ in implies a linear UV divergence of this integral which is cured by performing a Taylor expansion in pa up to the first derivative.

The first term of the Taylor expansion is odd in k, thus is zero, while the first derivative is

$$\int \frac{d^4k}{(2\pi)^4} \frac{-i\sum_\mu \gamma_\mu \sin(k_\mu)}{[(\omega' + b')_k + \frac{a^2m^2}{4}(\omega' - b')_k]} \sum_\rho p_\rho \frac{\partial}{\partial p_\rho a} \Big[\frac{\omega'_{p+k}}{[(\omega' + b')_{p+k} + \frac{a^2m^2}{4}(\omega' - b')_{p+k}]}\Big]_{p=0}$$

with

$$\frac{\partial}{\partial p_\rho a} \Big[\frac{\omega'_{p+k}}{[(\omega' + b')_{p+k} + \frac{a^2m^2}{4}(\omega' - b')_{p+k}]}\Big]_{p=0} = \frac{1}{[(\omega' + b')_k + \frac{a^2m^2}{4}(\omega' - b')_k]^2}$$
$$\times \Big(\frac{2}{\omega'_k} \sum_{\mu \neq \rho} \sin^2(\frac{k_\mu}{2}) \sin(k_\rho)(2\sum_\nu \sin^2(\frac{k_\nu}{2}) - 1) - \omega'_k \sin(k_\rho)\Big)$$

plus terms proportional to a^2m^2 which do not contribute in the limit $a \to 0$ while in the denominator the term proportional to a^2m^2 must be kept.

The last term of this derivative inserted in the expression produces a $\log(a^2m^2)$ divergence in the limit $a \to 0$.

Including the external leg factors the fermion two point function becomes

$$\langle \chi\bar{\chi}\rangle^{(2)}(p) = \frac{(i\not{p} - m)}{(p^2+m^2)} C_2 i\not{p} \frac{(i\not{p} - m)}{(p^2+m^2)}$$

where

$$C_2 = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{\omega'_k \sin^2(k_\rho)}{[(\omega+b)'_k + \frac{a^2m^2}{4}(\omega-b)'_k]^3} + C_{2f}$$

and C_{2f} is a finite number.

For the scalar two point function the integral to calculate is

$$\int \frac{d^4k}{(2\pi)^4}\Big\{\frac{m^2}{2}\frac{\omega'_k}{[(\omega'+b')_k+\frac{a^2m^2}{4}(\omega'-b')_k]}\frac{\omega'_{p+k}}{[(\omega'+b')_{p+k}+\frac{a^2m^2}{4}(\omega'-b')_{p+k}]}$$
$$-\frac{1}{a^2}\frac{1}{(\omega'+b')_k+\frac{a^2m^2}{4}(\omega'-b')_k}\frac{\omega'_{p+k}}{(\omega'+b')_{p+k}+\frac{a^2m^2}{4}(\omega'-b')_{p+k}}$$
$$\times\Big(\frac{\sum_\mu \sin^2(k_\mu)}{\omega'_k}-\frac{\sum_\mu \sin(k_\mu)\sin(k_\mu+ap_\mu)}{\omega'_{p+k}}\Big)\Big\}\,.$$

The first term can be evaluated directly at $pa=0$.

For the second we need a Taylor expansion up to the second derivative in pa due to the factor $1/a^2$.

We first concentrate on the latter term. It vanishes at $pa=0$ and moreover its first derivative is odd in k and therefore also this term of the expansion vanishes. We can rewrite

$$\int \frac{d^4k}{(2\pi)^4}\Big\{\frac{m^2}{2}\frac{(\omega'_k)^2}{[(\omega'+b')_k+\frac{a^2m^2}{4}(\omega'-b')_k]^2}$$
$$-\frac{1}{(\omega'+b')_k+\frac{a^2m^2}{4}(\omega'-b')_k}\frac{1}{2}\sum_{\rho\sigma}p_\rho p_\sigma\frac{\partial^2}{\partial p_\rho a\partial p_\sigma a}\Big[\frac{\omega'_{p+k}}{(\omega'+b')_{p+k}+\frac{a^2m^2}{4}(\omega'-b')_{p+k}}$$
$$\times\Big(\frac{\sum_\mu \sin^2(k_\mu)}{\omega'_k}-\frac{\sum_\mu \sin(k_\mu)\sin(k_\mu+ap_\mu)}{\omega'_{p+k}}\Big)\Big]_{p=0}\Big\}\,.$$

There are two contributions coming from the second derivative: one is the product of the derivative used before and

$$\frac{\partial}{\partial p_\sigma a}\frac{\sum_\mu \sin(k_\mu)\sin(k_\mu+ap_\mu)}{\omega'_{p+k}}\Big|_{p=0}=\frac{\sin(k_\sigma)\cos(k_\sigma)}{\omega'_k}$$
$$-\frac{2\sum_\mu \sin^2(k_\mu)\sum_{\nu\neq\sigma}\sin^2(\frac{k_\nu}{2})\sin(k_\sigma)}{(\omega'_k)^3}$$

and produces a $\log(a^2m^2)$. The rest is finite in the limit $a\to 0$.

Including the external leg factors, the two point function reads

$$D_2\langle AA\rangle^{(2)}(p) = i\ \not{p}\frac{1}{(p^2+m^2)}\frac{1}{(p^2+m^2)}(\frac{1}{2}C_3 m^2 - C_1 p^2)$$

where

$$C_3 = g^2 \int \frac{d^4k}{(2\pi)^4}\frac{(\omega'_k)^2}{[(\omega'+b')_k + \frac{a^2m^2}{4}(\omega'-b')_k]^2}\,,$$

$$C_1 = g^2 \int \frac{d^4k}{(2\pi)^4}\frac{\sin^2(k_\rho)\cos(k_\rho)}{[(\omega'+b')_k + \frac{a^2m^2}{4}(\omega'-b')_k]^3} + C_{1f}$$

and C_{1f} is a finite constant.

A similar analysis applied as before gives

$$\langle FA\rangle^{(2)}(p) = m\frac{1}{(p^2+m^2)}\frac{1}{(p^2+m^2)}(\frac{1}{2}C_3 + C_1)p^2 \,.$$

The continuum limit of the two point function containing the operator R are

$$\langle R^{(1)}A\rangle^{(1)}(p) = m\frac{(i\not p - m)}{(p^2+m^2)}\frac{1}{(p^2+m^2)}(C_2 - \frac{1}{2}C_3)i\not p$$

and

$$\langle R^{(2)}A\rangle^{(0)}(p) = \frac{(i\not p - m)}{(p^2+m^2)}\frac{1}{(p^2+m^2)}(C_2 - C_1)p^2 \,,$$

The combinations $C_2 - C_1$ and $C_2 - \frac{1}{2}C_3$ are two (different) finite numbers. Indeed, the $\log(a^2m^2)$ contributions cancels out in these combinations.

In the formulation of Fujikawa (without the R), the two-point functions of A, F and χ have the same logarithmic divergent parts.

Substituting all terms in the WTi with the corresponding signs one have

$$
\begin{aligned}
&\frac{(i\not{p}-m)}{(p^2+m^2)}(i\not{p}C_2)\frac{(i\not{p}-m)}{(p^2+m^2)} \\
&-\frac{i\not{p}}{(p^2+m^2)}(\frac{1}{2}m^2C_3-p^2C_1)\frac{1}{(p^2+m^2)} \\
&-\frac{m}{(p^2+m^2)}(C_1+\frac{1}{2}C_3)p^2\frac{1}{(p^2+m^2)} \\
&+\frac{(i\not{p}-m)}{(p^2+m^2)}(i\not{p}m)(C_2-\frac{1}{2}C_3)\frac{1}{(p^2+m^2)} \\
&+\frac{(i\not{p}-m)}{(p^2+m^2)}(C_2-C_1)p^2\frac{1}{(p^2+m^2)} = 0\,.
\end{aligned}
$$

Notice that the term $\langle RA\rangle$ above is essential to recover the WTi also in the continuum limit!!

Role of the operator R

- Thanks to the exactness of WTi it is always possible to write the two point function $\langle RA\rangle^{(2)}$ as a suitable combination of the other three two point functions involved in this WTi.

In particular, in the continuum limit one can write

$$\langle RA\rangle = \frac{i\not{p}-m}{p^2+m^2} i\not{p}\delta_1 \frac{i\not{p}-m}{p^2+m^2} + i\not{p}\frac{1}{p^2+m^2}(\delta_2 p^2+\delta_3 m^2)\frac{1}{p^2+m^2} - \frac{m}{p^2+m^2}(\delta_2-\delta_3)p^2\frac{1}{p^2+m^2}$$

where

$$\delta_1 = \frac{1}{2}C_3 - C_2 - \delta_3\,, \qquad \delta_2 = \frac{1}{2}C_3 - C_1 - \delta_3\,,$$

and the constant δ_3 is arbitrary.

Then in the continumm limit one can rewrite the WTi as the supersymmetric continuum WTi

$$\langle\chi\bar{\chi}\rangle_R^{(2)} - i\not{p}\langle AA\rangle_R^{(2)} - \langle FA\rangle_R^{(2)} = 0$$

with

$$\langle\chi\bar{\chi}\rangle_R^{(2)} \equiv \langle\chi\bar{\chi}\rangle^{(2)} + \frac{i\not{p} - m}{p^2+m^2} i\not{p}\delta_1 \frac{i\not{p} - m}{p^2+m^2}$$

$$\langle AA\rangle_R^{(2)} \equiv \langle AA\rangle_{(2)} - \frac{1}{p^2+m^2}(\delta_2 p^2 + \delta_3 m^2)\frac{1}{p^2+m^2}$$

$$\langle FA\rangle_R^{(2)} \equiv \langle FA\rangle^{(2)} + \frac{m}{p^2+m^2}(\delta_2 - \delta_3)p^2\frac{1}{p^2+m^2}$$

Writing these two point functions in terms of the 1PI vertex functions

$$\langle\chi\bar{\chi}\rangle^{(2)} = \frac{i\not{p} - m}{p^2+m^2}\Sigma_{\chi\bar{\chi}}^{(2)}\frac{i\not{p} - m}{p^2+m^2},$$

$$\langle AA\rangle^{(2)} = -\frac{1}{p^2+m^2}(\Sigma_{AA}^{(2)} + m^2\Sigma_{FF}^{(2)})\frac{1}{p^2+m^2},$$

$$\langle FA\rangle^{(2)} = \frac{1}{p^2+m^2}(\Sigma_{AA}^{(2)} - p^2\Sigma_{FF}^{(2)})\frac{m}{p^2+m^2},$$

The lattice contribution to these 1PI vertices in the continuum limit reads

$$\Sigma_{\chi\bar{\chi}}^{(2)} = i\not{p}C_2, \qquad \Sigma_{AA}^{(2)} = p^2 C_1, \qquad \Sigma_{FF}^{(2)} = -\frac{1}{2}C_3.$$

Moreover, one has

$$\Sigma^{(2)}_{\chi\bar{\chi}\,R} \equiv \Sigma^{(2)}_{\chi\bar{\chi}} + i \not{p}\delta_1 = i \not{p}(\frac{C_3}{2} - \delta_3) \equiv -Z_\chi i \not{p}$$

$$\Sigma^{(2)}_{AA\,R} \equiv \Sigma^{(2)}_{AA} + p^2\delta_2 = p^2(\frac{C_3}{2} - \delta_3) \equiv -Z_A p^2$$

$$\Sigma^{(2)}_{FF\,R} \equiv \Sigma^{(2)}_{FF} + \delta_3 = -(\frac{C_3}{2} - \delta_3) \equiv Z_F$$

with

$$Z_\chi = Z_A = Z_F = -(\frac{C_3}{2} - \delta_3)\,.$$

In Fujikawa's work it was shown that the one-loop corrections to the two-point function of A, F and χ differ by finite quantities. Our construction shows that if one redefines the 1PI vertices as in the wave function renormalization factors become equal.

This is an important consequence of the exact lattice supersymmetry we have introduced and of the WTi derived from this symmetry: leads to restoration of supersymmetry in the continuum limit with equal renormalization wave function for the scalar and fermion fields.

Zero modes in the action and R

The fermionic kinetic term in the action

$$(1 - \frac{a}{2}D_1)^{-1} D_2$$

needs a careful study at the border of the Brillouin zone. Let us take

$$p_\mu = (0, 0, 0, \frac{\pi - \epsilon}{a})$$

and study the limit $\epsilon \to 0$. We have that

$$b = \frac{1}{a}[1 - \frac{\epsilon^2}{2} + \mathcal{O}(\epsilon^4)]\,, \qquad \omega = \frac{1}{a}\,.$$

The we have

$$(1 - \frac{a}{2}D_1)^{-1} = \frac{4}{\epsilon^2}[1 + \mathcal{O}(\epsilon^2)]\,, \qquad D_2 = i\gamma_4 \frac{\epsilon}{a}[1 + \mathcal{O}(\epsilon^2)]\,.$$

Finally, the fermionic kinetic operator behaves as

$$(1 - \frac{a}{2}D_1)^{-1} D_2 = 4i\gamma_4 \frac{1}{a\epsilon}[1 + \mathcal{O}(\epsilon^2)]$$

Thus in the limit $\epsilon \to 0$ with fixed a the would be doubler becomes (infinitely) massive.

Similarly, one can check that this value of the momentum do not generate a pole in the bosonic propagators.

This analysis can be generalized to the other edges of the Brillouin zone. For instance,

$$p_\mu = (0, 0, \frac{\pi - \epsilon}{a}, \frac{\pi - \epsilon}{a})$$

we have

$$(1 - \frac{a}{2} D_1)^{-1} D_2 = i(\gamma_3 + \gamma_4) \frac{3}{a\epsilon}$$

which again becomes a massive mode when $\epsilon \to 0$ while a is kept fixed. All the rest of the would be zero modes behave in the same way.

Notice that the fermion propagator can be rewritten as

$$\langle \chi \bar{\chi} \rangle = -(D_2 - m(1 - \frac{a}{2} D_1))\,[\frac{2}{a} D_1 + m^2 (1 - \frac{a}{2} D_1)]^{-1}$$

which is clearly finite for $\epsilon \to 0$.

Summary

If the action is not invariant under a SUSY transformation the associated WTi contains a breaking term and the study of its continuum limit determines the counterterms nedeed to restore continuum SUSY (with fine tuning).

In the case of $N = 1$ SYM theory with Wilson fermions we presented a general procedure to determine the renormalization constants for the supercurrent and the mixing coeff.

The supercurrent not only mixes with T_μ but also a mixing with operators coming from the WTI appear. This extra mixing cancel out when the renormalized gluino mass is setting to zero and the on-shell condition is setting.

The value of Z_T to one-loop lattice PT is in agreement with numerical simulations.

If the action is invariant under a SUSY transformation the study of the associated WTi automatically lead to continuun SUSY without fine tuning of the renormalization constants, thus the procedure of renormalization is much easier.