Supersymmetry on the Lattice

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Outline

- Introduction
  Open questions of the SUSY Yang-Mills (SYM) dynamics
- Lattice formulation of $N = 1$ SYM theory
  - Curci-Veneziano: Wilson fermions
  - Kaplan-Schmaltz: Domain wall fermions
- Numerical simulations
  - Chiral symmetry breaking
  - Low-lying mass spectrum
  - SUSY Ward Identities (WIs)
- Exact supersymmetry on the lattice
- Wilson-Ginsparg fermions
- ...
- Conclusions and outlook

Introduction

SUSY conects fermions with bosons. In the case of perfet SUSY the boson and fermions are grouped in supermultiplet in which the members have the same masses.

- Possible solution to the hierarchy problem.
- Can incorporate the quantum gravity.
- ...

Since mass degeneracies are not observed in the present experiments → assume that SUSY is broken

It is generally assumed that the scale where SUSY becomes manifest is near the presently explored electroweak scale and that SUSY is spontaneously broken.

An actractive possibility for spontaneous symmetry breaking is to exploit non-perturbative mechanisms in SUSY gauge theories.

Lattice simulations (→ extract non perturbative dynamics in field theory) may be able to provide additional information and confirm the existing analytical calculations.

- Large number of parameters in the SM

Last plenary talk in SUSY on the lattice. Montvay '97. Review. Montvay '01.

The Model

The continuum action of $N = 1$ SYM and gauge group $SU(N_c)$ reads

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} \bar{\lambda}^a \gamma_\mu (\mathcal{D}_\mu \lambda)^a + m_{\tilde{g}} \bar{\lambda}^a \lambda^a \,,$$

$\lambda = \lambda^a T^a$ is a Majorana spinor in the adjoint representation of the gauge group that satisfies the Majorana condition

$$\bar{\lambda} = \lambda^T C \,, \qquad\qquad \lambda = C \bar{\lambda}^T$$

The gluon fields are represented by

$$\begin{aligned} A_\mu &= -ig A^a_\mu T^a \\ F_{\mu\nu} &= -ig F^a_{\mu\nu} T^a \\ \mathcal{D}_\mu \lambda^a &= \partial_\mu \lambda^a + g f_{abc} A^b_\mu \lambda^c \,. \end{aligned}$$

The action has for $m_{\tilde{g}} = 0$ a supersymmetry respect to the SUSY transformations.

The continuum SUSY transformations read

$$\begin{aligned} \delta A_\mu(x) &= -2g\bar{\lambda}(x)\gamma_\mu\varepsilon \\ \delta\lambda(x) &= -\frac{i}{g}\sigma_{\rho\tau}F_{\rho\tau}(x)\varepsilon \\ \delta\bar{\lambda}(x) &= \frac{i}{g}\bar{\varepsilon}\sigma_{\rho\tau}F_{\rho\tau}(x) \end{aligned}$$

where $\sigma_{\rho\tau} = \frac{i}{2}[\gamma_\rho, \gamma_\tau]$ and ε is a global Grassmann parameter with Majorana properties.

These transformations relate fermions and bosons.

The change of the action density is a total derivative

$$\delta\mathcal{L} = \partial_\mu\bar{j}_\mu\varepsilon = \varepsilon\partial_\mu j_\mu$$

where $j_\mu = -\frac{1}{2}S_\mu$ and the supercurrent is defined as

$$S_\mu \equiv -F^a_{\rho\tau}\sigma_{\rho\tau}\gamma_\mu\lambda^a\,.$$

Classically the Noether theorem is conserved

$$\partial_\mu S_\mu = 0\,,$$

(if the fields satisfy the eq. of motion). Furthermore, it fulfills a spin 3/2 constraint

$$\gamma_\mu S_\mu = 0\,.$$

Superfields

SUSY can be better understood in the language of superfields. For $N = 1$ SYM

$$S_{SUSY} = \Re\Big\{ \int d^4x\, d^2\theta\, \mathrm{Tr}(W^\alpha W_\alpha) \Big\}\,,$$

where $W(x, \theta, \bar\theta)_\alpha$ is the spinorial field strength superfield.

$$\begin{aligned} S_{SUSY} \;\; = \;\; & \int d^4x \mathrm{Tr}\Big\{ -\frac{1}{2} F^a_{\mu\nu} F^{\mu\nu a} + \frac{i}{2} F^a_{\mu\nu} \tilde{F}^{\mu\nu a} \\ & -i\psi^\alpha_w \sigma^\mu (D_\mu \bar\psi_w)^a + i(D_\mu \bar\psi_w)^a \bar\sigma^\mu \psi^a_w + D^2 \Big\}\,, \end{aligned}$$

Bagger & Wess '83
Fayet & Ferrara '77
Sohnius '85

$$F_{\mu\nu} \equiv -igF^a_{\mu\nu}(x)T^a\,, \;\; \tilde{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma a}(x) T^a\,.$$

The action includes the Θ-term

$$\tau \equiv \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$$

The action $N = 1$ SYM becomes

$$\frac{1}{4\pi}\Im\Big\{\tau\int d^4x\, d^2\theta\,\mathrm{Tr}(W^\alpha W_\alpha)\Big\} =$$
$$\frac{1}{g^2}\int d^4x\mathrm{Tr}\Big[-\frac{1}{2}F^a_{\mu\nu}F^{\mu\nu a} - i\psi^a_w\sigma^\mu(D_\mu\bar{\psi}_w)^a + i(D_\mu\bar{\psi}_w)^a\bar{\sigma}^\mu\psi^a_w$$
$$+D^2\Big] + \frac{\Theta}{16\pi^2}\int d^4x\mathrm{Tr}\Big[F^a_{\mu\nu}\tilde{F}^{\mu\nu a}\Big].$$

- Performing a Gaussian integration over the auxiliary field D
- going to Euclidean space
- setting Θ-parameter $= 0$

Recover the action for a SYM with a massless Majorana fermion in the adjoint representation.

SUSY WIs

One expect the existence of the renormalized supercurrent S_μ^R

$$\partial_\mu S_\mu^R = 2 m_R \chi_R$$

where

$$\chi_R = Z_\chi \chi \,, \qquad \chi \equiv \frac{1}{2} F_{\mu\nu}^a \sigma_{\mu\nu} \lambda^a \,.$$

m_R is the renormalized gluino mass.

- SUSY occurs for $m_R = 0$.
- The non-vanishing of m_R describes a **soft breaking** of SUSY.

Open questions of the SYM dynamics

The basic feature of SYM dynamics (similar to QCD):

- Confinement
- Spontaneous chiral symmetry breaking

  $$U_A(1)(R) \stackrel{anomaly}{\longrightarrow} Z_{2N_c} \stackrel{spontaneous}{\longrightarrow} Z_2$$

  Gluino condensation: $\langle \bar{\lambda}\lambda \rangle \neq 0$
- Low-lying mass spectrum
- SUSY Ward identities (anomaly?)

Casher & Shamir '99

This are **non-perturbative effects**
$\Longrightarrow$ Lattice formulation

Chiral symmetry breaking

Introducing a non zero gluino mass term

$$\mathcal{L}_{mass} = m_{\tilde{g}} \bar{\lambda}^a \lambda^a$$

breaks SUSY softly → Non-renormalization theorem and cancellation of divergencies are preserved

Girardello & Grisaru '82

In the massless case, the global chiral symmetry is $U(1)_\psi$

$$\lambda \to e^{-i\varphi\gamma_5}\lambda\,, \qquad \bar{\lambda} \to \bar{\lambda} e^{-i\varphi\gamma_5}\,.$$

It is anomalous

$$\partial_\mu J^5_\mu = \frac{N_c g^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}\,.$$

The anomaly leaves a Z_{2N_c} subgroup of $U(1)_\psi$ unbroken

These transformations are equivalent to

$$m_{\tilde{g}} \to m_{\tilde{g}} e^{-2i\varphi\gamma_5}\,, \quad \Theta_{SUSY} \to \Theta_{SUSY} - 2N_c\varphi$$

In the SUSY case $m_{\tilde{g}} = 0$, $U(1)_\phi$ symmetry is unbroken if

$$\varphi = \varphi_k \equiv \frac{k\pi}{N_c}\,, \qquad (k = 0, 1, \cdots, 2N_c - 1)$$

Z_{2N_c} is expected to be spontaneously broken to Z_2 by $\langle \bar{\lambda}\lambda \rangle \neq 0$

Novikov, Shifman, Vainshtein & Zakarov '85

Consequence of this spontaneous chiral symmetry breaking is

$\Longrightarrow$ First order phase transition at $m_{\tilde{g}} = 0$

$\Rightarrow$ Existence of N_c degenerate ground states with different orientations of the gluino condensate

$(k = 0, \cdots, N_c - 1)$

$$\langle \bar{\lambda}\lambda \rangle = c\Lambda^3 e^{\frac{2\pi i k}{N_c}}$$

Dependence of the gauge group

- **case $SU(2)$:** Two degenerate ground states with opposite signs of the gluino condensate. $\langle \bar{\lambda}\lambda \rangle < 0\,, \quad \langle \bar{\lambda}\lambda \rangle > 0$
- **case $SU(3)$:** There are (at least) three degenerate vacua at $k = k_c$

  fourth state: $< \lambda\lambda >= 0$

Kovner & Shifman '97

Magnitude of the gluino condensate

The calculation of the gluino condensate in $N = 1$ SYM theory. A puzzle!

Two approach in the literature for calculating (which gives differents results)

- Based on weak-coupling instanton (WCI) calculations

$$\langle \lambda\lambda \rangle = c\Lambda^3$$
$$\Lambda = M_{PV}\left(\frac{16\pi^2}{3N_c g^2}\right)^{1/3} \exp\left(-\frac{8\pi^2}{3N_c g^2}\right)$$

  with $c = 6$.

Affleck, Dine & Seiberg '83,'84,'85
Novikov, Shifman, Vainshtein & Zakharov '85
Shifman & Vainshtein '88

- Based on strong-coupling instanton (SCI) calculations with $c = \frac{4}{5}$

Novikov, Shifman, Vainshtein & Zakharov '83
Rossi & Veneziano '84
Amati, Rossi & Veneziano '84
Fuchs & Schmidt '86
Amati, Konishi, Meurice, Rossi & Veneziano '88

Serious doubt on the SCI calculation by showing that the cluster decomposition is invalid

Hollowood, Khoze, Lee & Mattis '99

Also, the addition of a so-called Kovner-Shifman chiral symmetric vacuum state $(\langle\lambda\lambda\rangle = 0)$

Kovner & Shifman '97
Shifman & Vainshtein '99

can not straighforwardly resolve this mismatch.

Hollowood, Khoze, Lee & Mattis '99

Using an instanton liquid picture gives qualitatively similar results and evidence for the condensate

Schäfer '00

Light hadron spectrum

An effective action for the low energy behavior of $N = 1$ SYM theory has been proposed

Veneziano & Yankielowicz '82

To construct the action they identify all degrees of freedom they expect to govern the low energy dynamics. (Gauge invariant and colorless composite fields)

$$F^a_{\mu\nu} F^a_{\mu\nu}$$
$$F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}$$
$$\bar{\lambda}^a \lambda^a$$
$$\sigma_{\mu\nu} F^a_{\mu\nu} \lambda^a$$

The first three operators are known in QCD while $\chi = \sigma_{\mu\nu} F^a_{\mu\nu} \lambda^a$ is a new type of composite operator formed by the gluino λ and the gauge field F.

The fields can be combined to form the chiral superfield

$$S(x,\theta) = \phi(x) + \sqrt{2}\theta\chi(x) + \theta\theta F(x)$$

$$\phi = \frac{\beta(g)}{2g}(\psi_w)^\alpha(\psi_w)_\alpha\,,\ \sqrt{2}\chi_\alpha = -\frac{\beta(g)}{2g}(-i(\psi_w)_\alpha D + (\sigma^{\mu\nu}\psi_w)_\alpha F_{\mu\nu})\,,$$
$$F = -\frac{\beta(g)}{g}\Big\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{i}{2}\psi_w\sigma^\mu\partial_\mu\bar{\psi}_w - \frac{i}{8}F_{\mu\nu}\varepsilon_{\mu\nu\rho\tau}F_{\rho\tau} + \frac{i}{2}\partial_\mu J^5_\mu + \frac{1}{2}D^2\Big\}\,.$$

The effective VY action is

$$\mathcal{L}_{eff} = \frac{1}{\alpha}(S^\dagger S)^{1/3}|_D + \gamma[(S\log\frac{S}{\mu^3} - S)|_F + h.c.]\,.$$

Expanding the effective action around its minimum, it is found the low-lying spectrum forming a supermultiplet of the Wess-Zumino type consist of

- A scalar boson $\phi = \bar{\lambda}^a \lambda^a$. In analogy with QCD. (The gluino is in the adjoint representation) $\rightarrow a - f_0$.

- A pseudoscalar boson $\phi_p = \bar{\lambda}^a \gamma_5 \lambda^a \rightarrow a - \eta\prime$.

- A massive Majorana fermion $\chi = \sigma_{\mu\nu} F^a_{\mu\nu} \lambda^a \rightarrow$ gluino-glue.

It is not clear why glueballs should be absent in the low-lying spectrum?

The introdution of a non-zero gluino mass breaks SUSY softly and leads to a splitting of the multiplet.

How the spectrum of glueballs, gluinoballs and gluino-glueballs are influenced by the soft SUSY breaking due to a non-zero gluino mass $m_{\tilde{g}} \neq 0$

$$\begin{aligned} M_{a-\eta'} &= N_c \alpha \Lambda + \frac{40\pi^2 |m_{\tilde{g}}|}{3N_c} \\ M_{a-\chi} &= N_c \alpha \Lambda + \frac{48\pi^2 |m_{\tilde{g}}|}{3N_c} \\ M_{a-f_0} &= N_c \alpha \Lambda + \frac{56\pi^2 |m_{\tilde{g}}|}{3N_c} \end{aligned}$$

Evans, Hsu & Schwetz '97

The range of applicability of the linear mass formulas $\rightarrow$ not known.

The question of how to include glueballs in the low energy spectrum

Farrar, Gabadadze & Schwetz '98

Introducing an extra term in the effective action $\Longrightarrow$ gives the dynamics for the glueballs.

For unbroken SUSY the masses of these two supermultiplets are not identical. The heavier supermultiplet corresponds to the VY multiplet. The lighter one contains

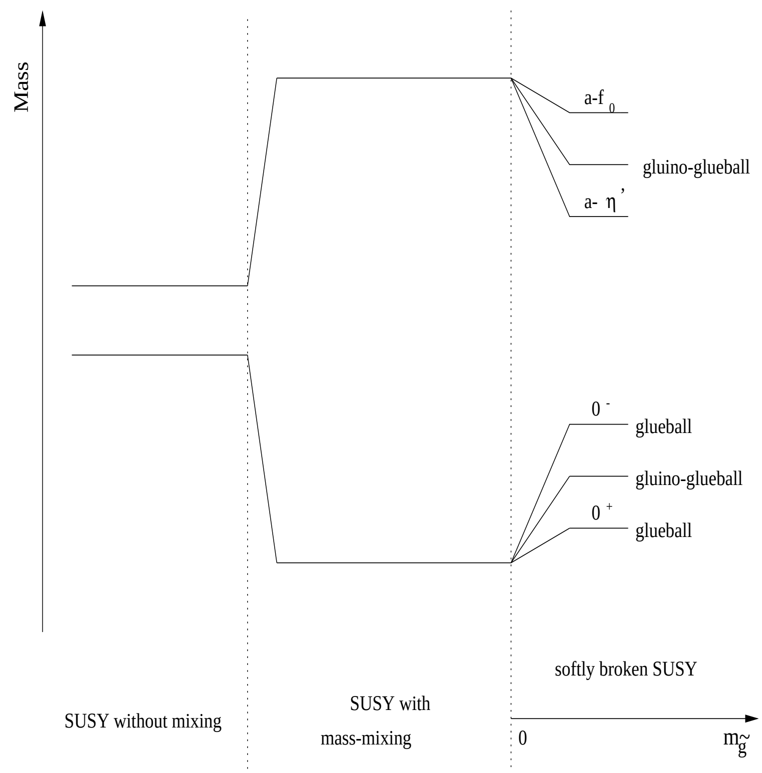
- In the low effective action of FGS $\rightarrow$ **non-zero mixing** between the states in the two light supermultiplets.
  - It can be a mixing between $a - f_0$ gluinoball and 0^+ glueball.

- A 0^+glueball $\approx F_{\mu\nu}F_{\mu\nu}$,

- A 0^-glueball $\approx F_{\mu\nu}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$,

- A gluino-glue ground state.

Masses


Mass
a-f 0
gluino-glueball
a- η '
0 -
glueball
gluino-glueball
0 +
glueball
softly broken SUSY
SUSY without mixing
SUSY with
mass-mixing
0
m g̃

Lattice formulation of SYM theory

Consider the problems of putting SUSY theory on the lattice

- The lattice regularized theory is not SUSY as the Poincaré invariance (a sector of the super-algebra) is lost.

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu$$

  Not a severe problem. Calculating at several lattice spacing a and then take the limit $a \to 0$. No fine tuning is needed.

- If there are scalar mass terms in the SUSY theory that break SUSY. Since this operators are relevant fine tuning is necessary to cancel their contributions.

- A naive regularization of fermions results in the doubling problem

  Nielsen & Ninomiya '81

  $\to$ wrong number of fermions and violation of the balance between bosons and fermions

  – The problem can be treated as in QCD. This is the case of $N = 1$ SYM.

Wilson fermions

Propose to give up manifest SUSY on the lattice and restore it in the continuum limit.

Curci & Veneziano '87

SUSY is broken by the lattice, by the Wilson term and a a soft breaking due to the gluino mass is present.

- SUSY is recovered in the continuum limit by tuning the bare parameters g and gluino mass $m_{\tilde{g}}$ to the SUSY point.

- The chiral and SUSY limit can be recovered simultaneously at $m_{\tilde{g}} = 0$.

Wilson fermions

The Curci and Veneziano action reads

$$S = S_G + S_F,$$

$$S_G = \frac{\beta}{2} \sum_x \sum_{\mu\nu} \left(1 - \frac{1}{N_c} \mathrm{Re}\,\mathrm{Tr}\, U_{\mu\nu}(x)\right),$$

and $\beta \equiv 2N_c/g_0^2$ correspond to the bare gauge coupling.

$$\begin{aligned} S_F &= \mathrm{Tr}\left\{\frac{1}{2a}\Big(\bar\lambda(x)(\gamma_\mu - r)U_\mu^\dagger(x)\lambda(x+a\hat\mu)U_\mu(x)\right. \\ &\quad -\bar\lambda(x+a\hat\mu)(\gamma_\mu + r)U_\mu(x)\lambda(x)U_\mu^\dagger(x)\Big) \\ &\quad \left. + \left(m_0 + \frac{4r}{a}\right)\bar\lambda(x)\lambda(x)\right\}. \end{aligned}$$

The Grassmann variables λ and $\bar\lambda$ are not independent

$$\bar\lambda = \lambda^T C\,, \qquad \lambda = C\bar\lambda^T\,.$$

Monte Carlo simulations : A different parametrization is used. The hopping parameter k is

$$k = \frac{1}{2(4 + m_0 a)}$$

The lattice Wilson fermion action

$$S_F \equiv \frac{1}{2}\bar{\lambda} Q \lambda \equiv \frac{1}{2}\sum_x \Big\{ \bar{\lambda}^a_x \lambda^a_x -$$

$$k \sum_{\mu=1}^{4} \left[\bar{\lambda}^a_{x+\hat{\mu}} V_{ab,x\mu}(1+\gamma_\mu)\lambda^b_x + \bar{\lambda}^a V^T_{ab,x\mu}(1-\gamma_\mu)\lambda^b_{x+\hat{\mu}} \right] \Big\}$$

with the adjoint link $V_{ab,x\mu}(x)$ in the adjoint representation

$$\begin{aligned} V_{ab,x\mu} &\equiv V_{ab,x\mu}[U] \equiv \\ &\equiv 2\mathrm{Tr}(U^\dagger_{x\mu} T_a U_{x\mu} T_b) = V^*_{ab,x\mu} = V^{-1T}_{ab,x\mu} . \end{aligned}$$

The path integral over the Majorana fermions gives the Pfaffian

$$\int [d\lambda] e^{-\frac{1}{2}\bar{\lambda} Q \lambda} = \int [d\lambda] e^{-\frac{1}{2}\lambda^T C Q \lambda} = Pf(M) = \underbrace{\pm}_{\text{sign}} \sqrt{det\, Q} .$$

where $M \equiv CQ = -M^T$ is an antisymmetric matrix.

Algorithm for numerical simulations

The effective action is

$$S_{CV} = \beta \sum_{pl} \left(1 - \frac{1}{2}\mathrm{Tr} U_{pl}\right) - \frac{1}{2} \log \det Q[U] \,.$$

Curci & Veneziano '87

The ommited sign of the Pfaffian can be taken into account by the reweighting formula

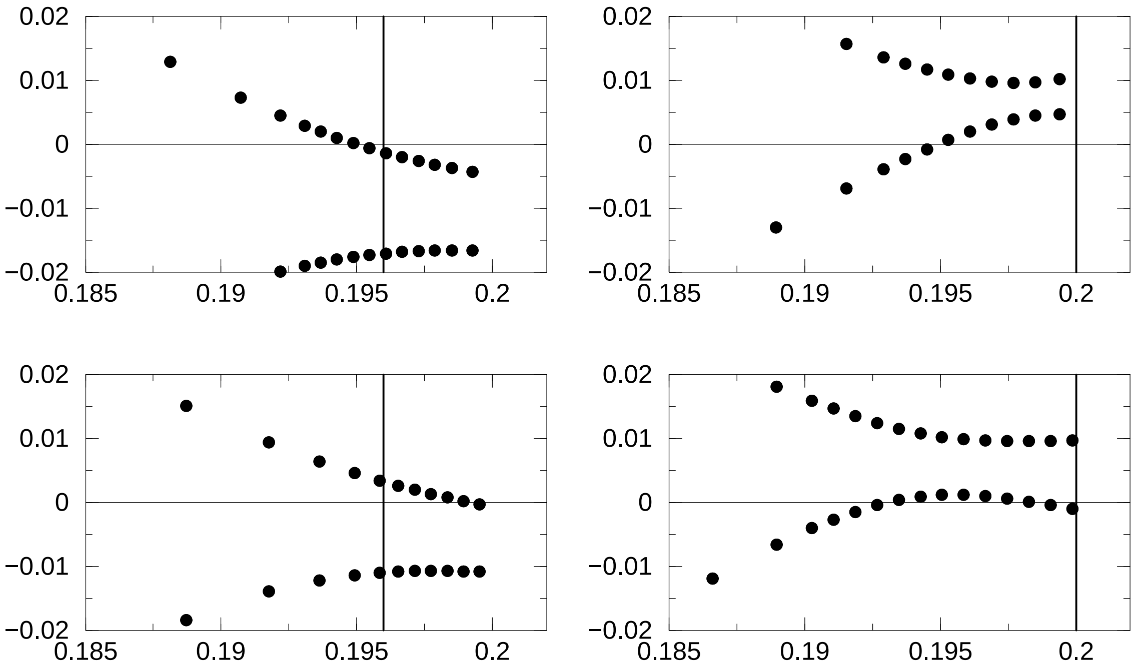
$$\langle \mathcal{O} \rangle = \frac{\left\langle \mathcal{O}\, \mathrm{sign} Pf(M) \right\rangle_{CV}}{\left\langle \mathrm{sign} Pf(M) \right\rangle_{CV}}$$

may rise to the **sign problem**

Spectral flow method: for the sign.

$$|Pf(M)| = \prod_{i=1}^{\Omega/2} |\tilde{\lambda}_i| \,, \quad \Longrightarrow Pf(M) = \prod_{i=1}^{\Omega/2} \tilde{\lambda}_i \,.$$

If the value of an eigenvalue $\tilde{\lambda}_i$ changes sign, the sign of $Pf(M)$ has to change too.



The spectral flow of the hermitian fermion matrix $\tilde{Q}$ for a configuration on $6^3 \times 12$ at $\beta = 2.3$. The value of k in the simulation correspond to the vertical line. DESY-Münster Collaboration '99

For $k < k_c \longrightarrow$ no serious sign problems!

The factor $\frac{1}{2} \Longrightarrow$ Majorana fermions mean a flavor number $N_f = \frac{1}{2}$.

Details on how from the path integral it can be obtained the gluino propagator and the n-point correlation functions

Montvay '96

- Simulate with *hybrid molecular dynamics algorithm* (HMD)

  Gottlieb, Liu, Toussaint, Renken & Sugar '87

  (applicable to any number of flavors).

  – Checked for the CV action for $N = 1$ $SU(2)$ SYM, for small lattices ($4^3 \times 8$)

    Donini & Guagnelli '96

- Simulate with non-even numbers of flavors based on the *multi-bosonic algorithm*

  Lüscher '94

  A two-step variant using a *noisy correction step*

  Kennedy, Kuti, Meyer & Pendleton '88

has been develop in the *Two-step multibosonic algorithm* (TSMB)

$$\mathcal{P}_1(x)\mathcal{P}_2(s) \approx x^{-N_f/2}$$

$\mathcal{P}_1(x)$ (low order, is replaced by the partition function of a small number of boson fields) and $\mathcal{P}_2(x)$ (higher order, is used in the acceptance rate.

Montvay '96,'98

- Unquenched results using TSMB $\rightarrow$ DESY-Münster-Roma Collaboration (the first large scale numerical simulation on SYM theory).

  * For $SU(2)$

    Kirchner, Luckmann, Montvay, Spanderen & Westphalen '99
    Campos, F., Kirchner, Luckmann, Montvay, Münster, Spanderen & Westphalen '99
    Farchioni, F., Galla, Gebert, Kirchner, Montvay, Münster & Vladikas '02

  * For $SU(3)$

    F., Kirchner, Luckmann, Montvay & Münster '00

Quenched interesting results (pioneers works)

Koutsoumbas & Montvay '97
Donini, Guagnelli, Hernandez & Vladikas '98
Donini, Gabrielli & Gavela '99

Study of large N corrections to the strong-coupling behavior

Gabrielli, Gonzalez-Arroyo & Pena '00
Gonzalez-Arroyo & Pena '00

Quenched approx. is exact at this order!
Consistent with the quenched results of Donini et al.

Domain wall fermions

A new lattice fermion regulator. Very nice innovation. Application of DWF in SUSY theories

Neuberger '98
Kaplan & Schmaltz '00

Monte Carlo simulation for $N = 1$ $SU(2)$ SYM with DWF

Fleming, Kogut & Vranas '01
Fleming '01

DWF were introduced in

Kaplan '92,'93

with further developed in

Narayanan & Neuberger '93,'94,'95
Shamir '93
Furman & Shamir '95

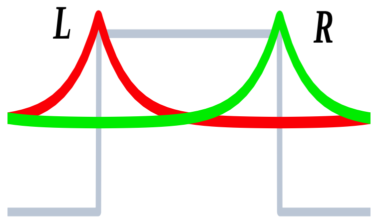
For a review in DFW in SUSY, please see plenary talk by Vranas, Lattice 2000.

Difficulties in using Wilson fermions.

- Need to fine tuning. The Wilson term breaks chiral symmetry
- The Pfaffian. Is not positive definite at finite lattice spacing.

DWF are defined extending space-time to five dimensions.

L_s is the size of the fifth dimension.



(fig. from Kaplan '00)

In the limit $L_s \to \infty$ chiral symmetry is exact, even at finite lattice spacing.

- There is not need for fine tuning.

The domain wall action is

$$S = S_G(U) + S_F(\Psi, U) + S_{PV}(\Phi, U)$$

$$S_F = -\sum_{x,x',s,s'} \bar{\Psi}_{x,s}(D_F)_{x,s;x',s'}\Psi_{x',s'}$$

D_F is the DWF Dirac operator

$$(D_F)_{x,s;x',s'} = \delta_{s,s'} \not{D}_{x,x'} + \not{D}^{\perp}_{s,s'}\delta_{x,x'}$$

In an s-block form

$$D_F = \begin{pmatrix} 1+D & -P_L & 0 & 0 & \cdots & 0 & 0 & m_f P_R \\ -P_R & 1+D & -P_L & 0 & \cdots & 0 & 0 & 0 \\ 0 & -P_R & 1+D & -P_L & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -P_R & 1+D & -P_L \\ m_f P_L & 0 & 0 & 0 & \cdots & 0 & -P_R & 1+D \end{pmatrix}$$

$$P_{R,L} = \frac{1 \pm \gamma_5}{2}$$

$$\begin{aligned} D\!\!\!/_{ax,a'x'} &= (4 - m_0)\delta_{xx'} \\ &-\frac{1}{2}\sum_{\mu=1}^{4}\left[\delta_{x,x'+\hat{\mu}} V_{aa',x'\mu}(1+\gamma_\mu) + \delta_{x+\hat{\mu},x'} V^T_{aa',x\mu}(1-\gamma_\mu)\right] \end{aligned}$$

m_0 is the domain wall height or five-dimensional mass that controls the number of flavors.

The fermion field $\bar{\Psi}$ is related to Ψ by the equivalent Majorana condition for this 5-dimensional theory

$$\bar{\Psi} = \Psi^T C R_5$$

Kaplan & Schmaltz '00

where R_5 is the reflection operator and $C = \gamma_0\gamma_2$.

The gluino action is write as

$$S_F = -\sum_{x,x',s,s'} \Psi^T (M_F)_{x,s;x',s'} \Psi_{x',s}$$

Where the antisymmetric fermion matrix is

$$M_F = CR_5 D_F\,,$$

Neuberger '98

The fermionic path integral gives the Pfaffian

$$\int [d\Psi] e^{-S_F} = Pf(M_F) = \sqrt{det(D_F)}\,.$$

Positive for $m_f > 0$.

This is an advantage of the DWF beside the good chiral properties for $m_f = 0$

The Pauli-Villars action S_{PV} is designed to cancel contributions of the heavy bulk fermions $L_s - 1$.

Narayanan & Neuberger '93,'94,'95

S_{PV} substract the L_s heavy particles and the light domain wall particle is double regularized.

Neuberger '98

The effective action is

$$\begin{aligned} S_{KS} &= \beta \sum_{pl} \left(1 - \frac{1}{2}\mathrm{Tr} U_{pl}\right) - \frac{1}{2}\log\det D_F[U] \\ &+ \frac{1}{2}\log\det D_F[m_f = 1; U]\,. \end{aligned}$$

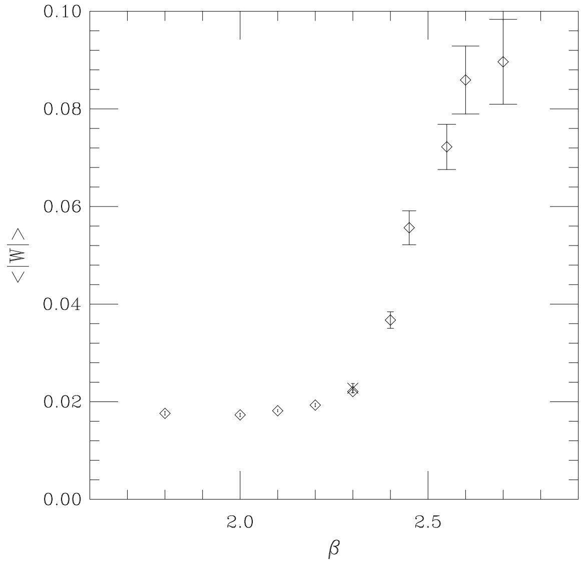
Difficulties in using DWF.

- 2 extra parameters in DWF: L_s and m_0.

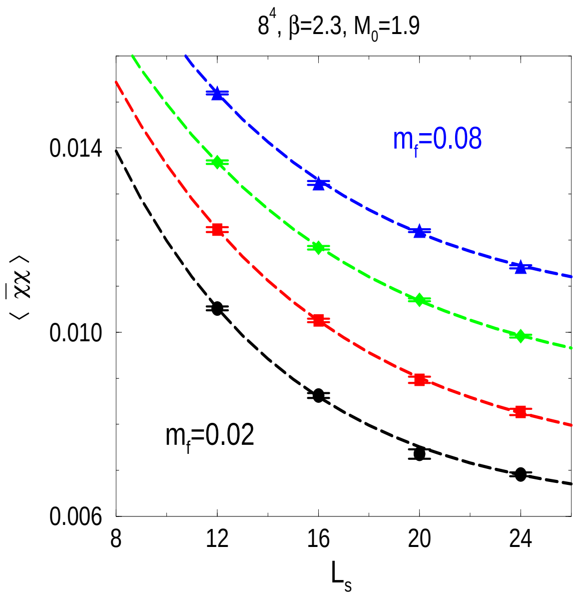
  $$m_{eff} = m_0(2-m_0)[m_f+(1-m_0)^{L_s}]\,, \quad 0 < m_0 < 2$$

- The two chiralities do not decouple $\rightarrow$ no restoration of chiral symmetry. (Need large values of L_s)

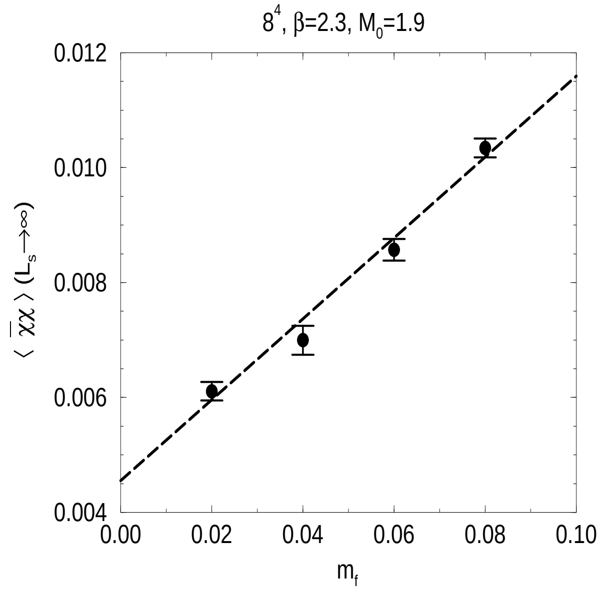
- Harder to simulate than QCD (with Wilson fermions easier to simulate than QCD)



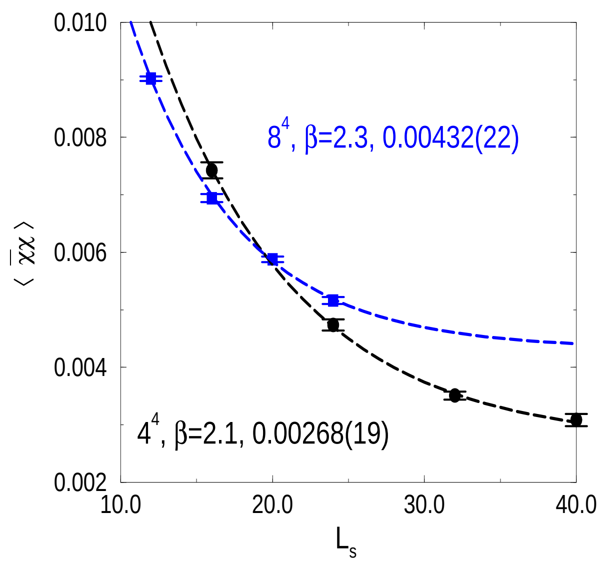
Wilson line on a 8^4 lattice. Diamonds are quenched and the cross is dynamical with $L_s = 24$ and $m_f = 0$.



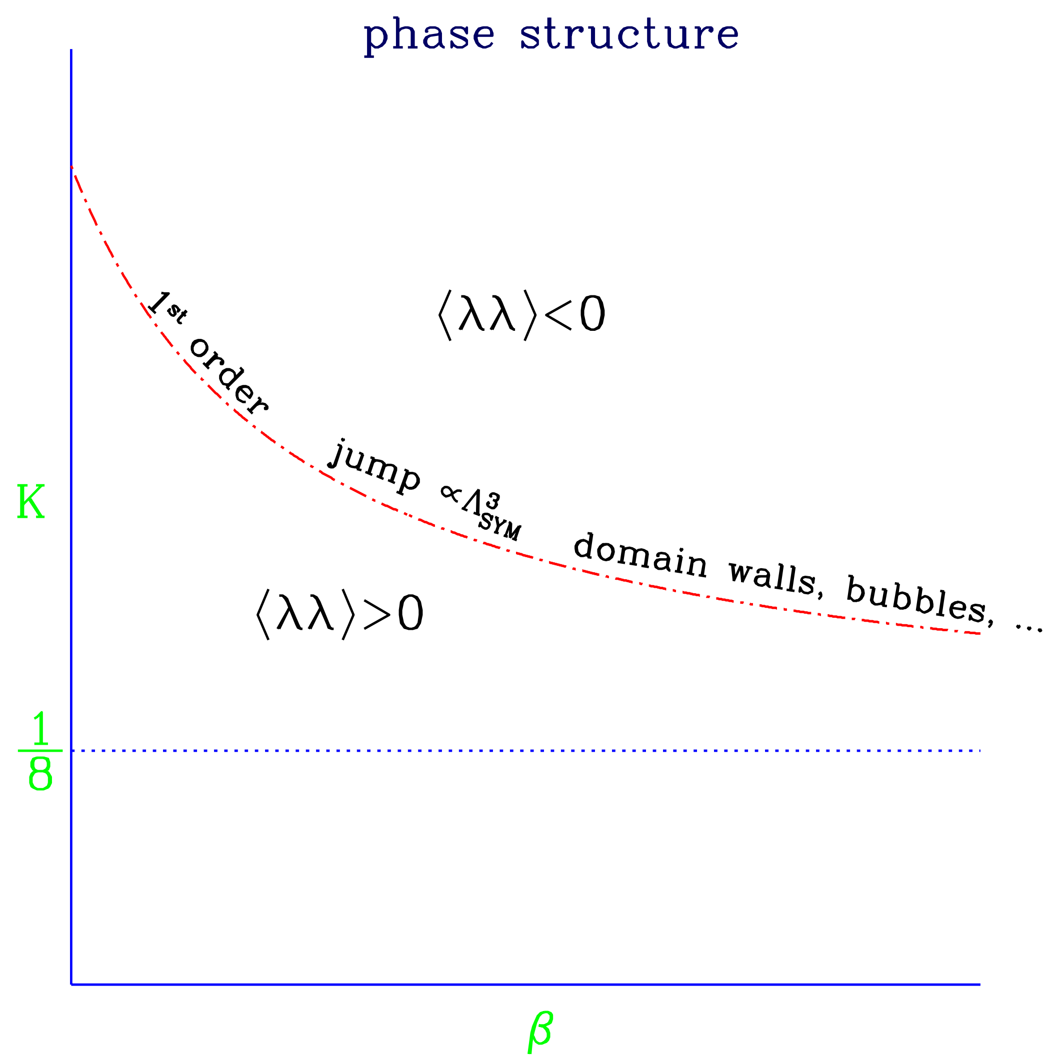
Gluino condensate vs L_s with $\beta = 2.3$ and $m_f = 0.02, 0.04, 0.06, 0.08$. (Fleming, Kogut, Vranas '00)



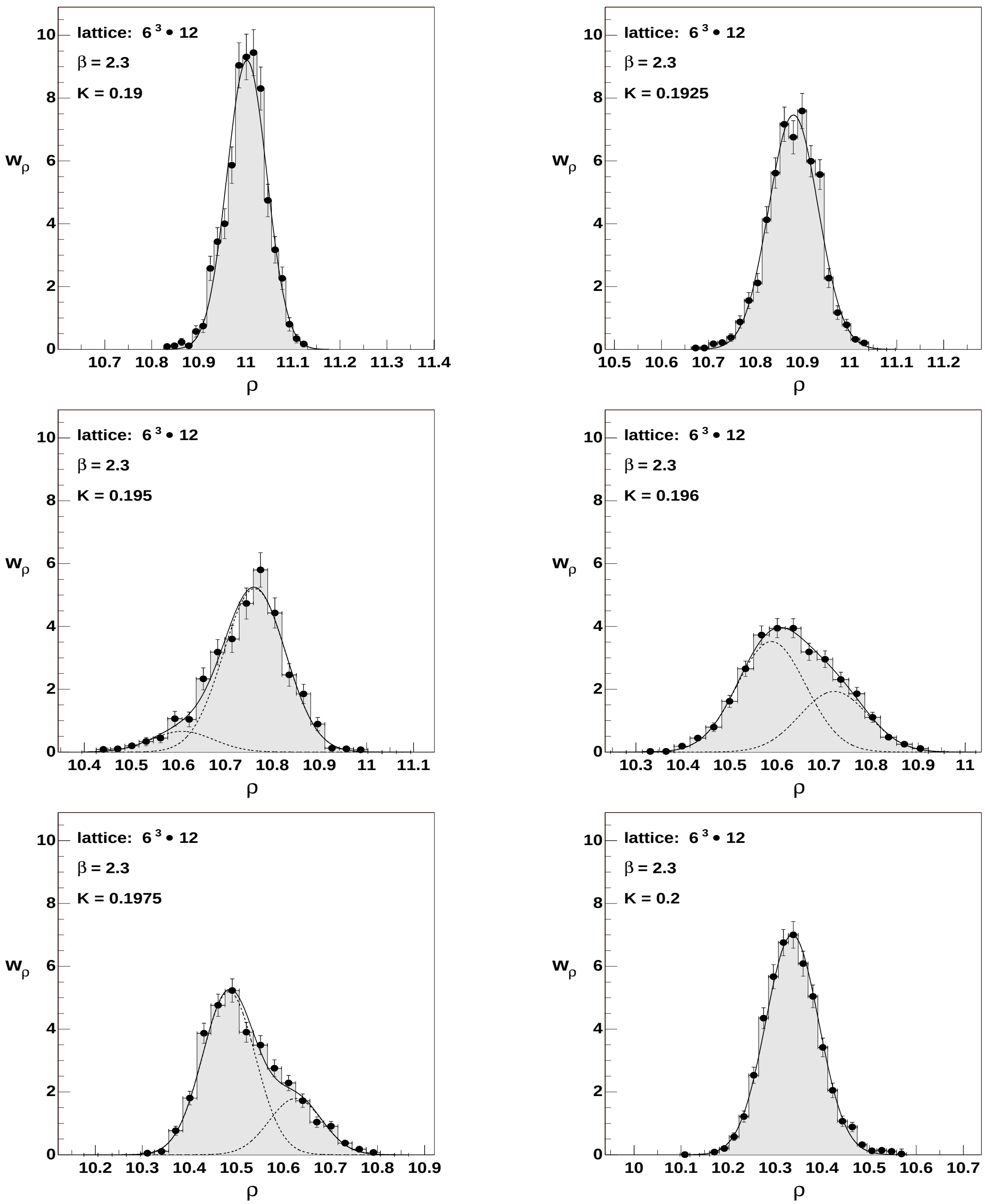
Extrapoleted gluino condesante to $L_s \to \infty$ limit vs m_f and linear fit to $m_f \to 0$ limit.



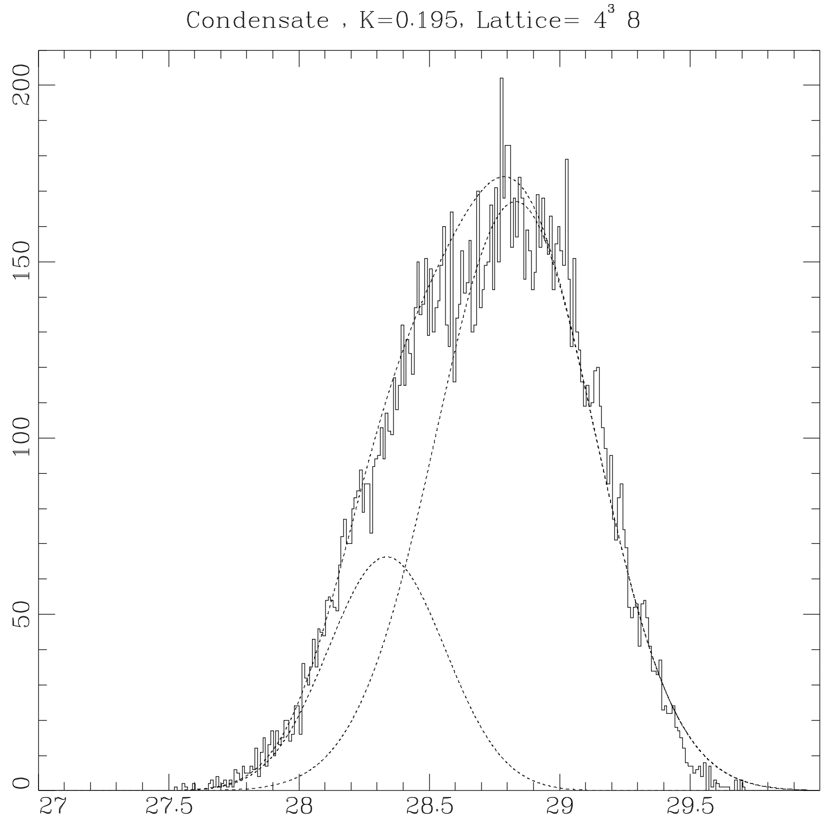
Dynamical gluino condensate at $m_f = 0$ vs L_s on two different lattices. (Fleming, Kogut, Vranas '00)

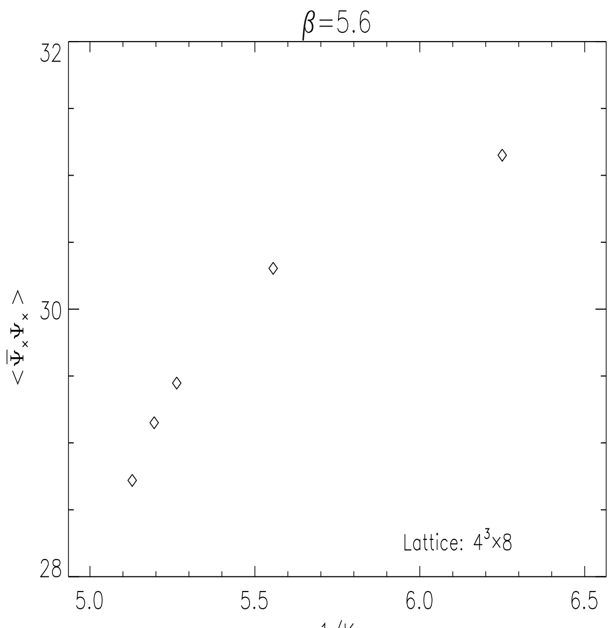


Expected phase structure of SYM in the (β, k) plane. Dashed line $k = k_c(\beta)$ is a first-order phase transition (or cross-over) at $m_{\tilde{g}} = 0$.



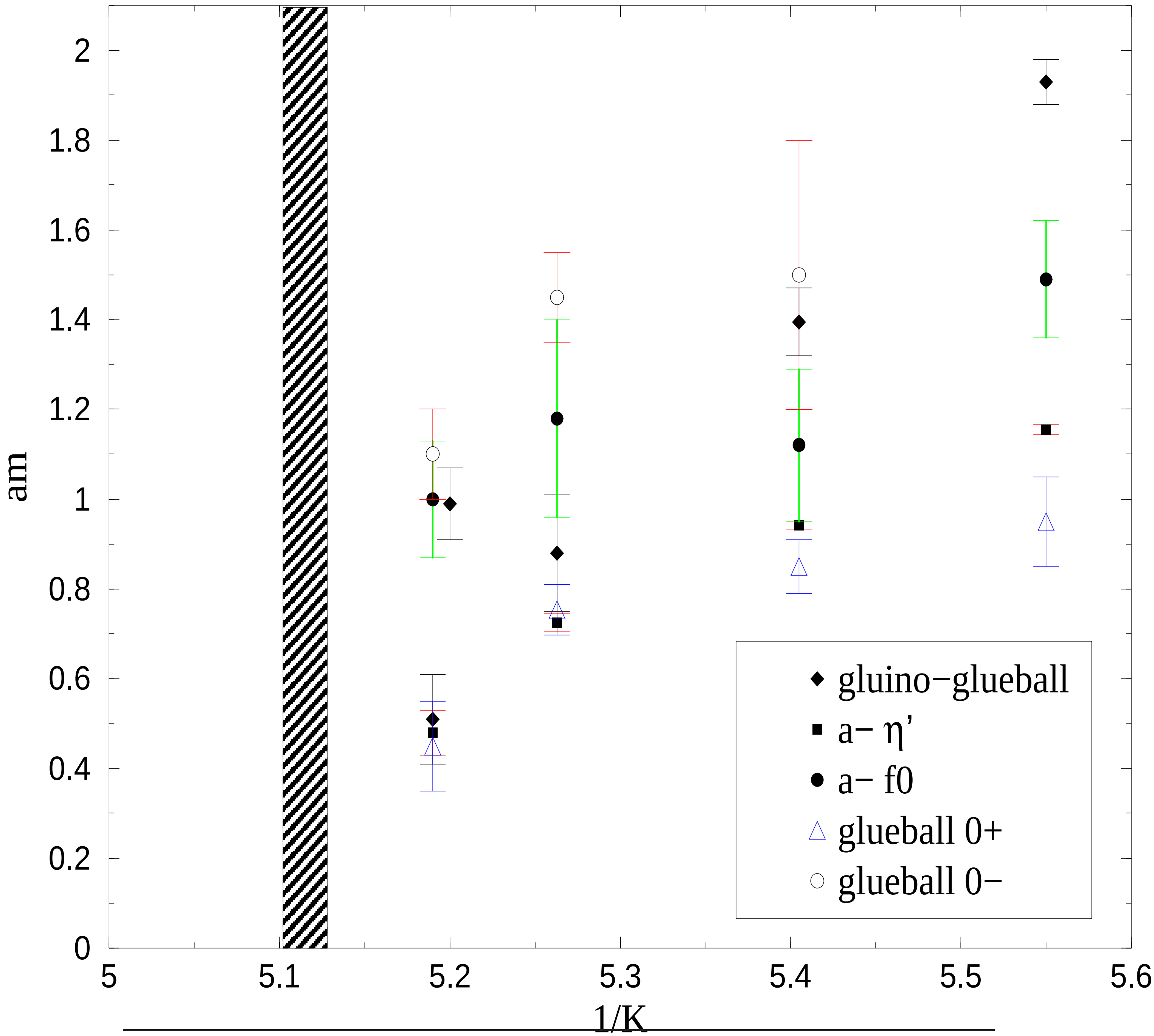
Distribution of the gluino condensate for different k values at $\beta = 2.3$ and lattice size $6^3 \times 12$, $k_c = (0.1955 \pm 0.0005)$ and $N_c = 2$. (DESY-Münster Collaboration, '99)





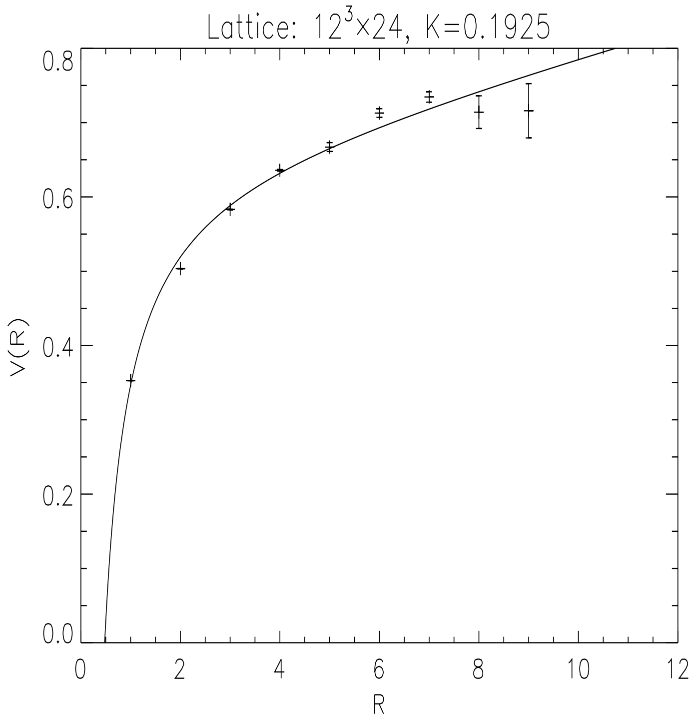
Distribution of the gluino condensate for $k = 0.195$ at $\beta = 5.6$ and lattice size $4^3 \times 8$ and $N_c = 3$. (DESY-Münster Collaboration, '00)

Light hadron spectrum in lattice units (DESY-Münster Collaboration '00, '02). $k_c = (0.1955 \pm 0.0005)$. Lattice size $12^3 \times 24$. See Peetz's Poster, this conference.



The string tension σ is obtained by fitting the potential

$$V(R) = V_0 - \frac{\alpha}{R} + \sigma R\,.$$



The static quark potential $V(R)$. (DESY-Münster Collaboration, '99)

$a\sqrt{\sigma} = 0.17 \pm 0.01$

$L = 12a \quad \sqrt{\sigma_{QCD}} = 0.44 GeV \quad \Longrightarrow L = 1 fm$

Seems to be not large enough for finite volume effects to be absent! Also see poster by Peetz, new value for lattice size $16^3 \times 32$.

SUSY WIs on the lattice

Another independent way to study the SUSY limit (chiral limit) can be also defined using the SUSY Ward identities (WIs).

Additional breaking terms in the lattice arise from the explicit breaking of SUSY.

The SUSY limit is defined to be the point in parameter space in which this breaking terms vanish and the WIs takes its continuum limit.

Nevertheless SUSY is not fullfilled on the lattice one might still define some SUSY transformations. One choice is

$$\begin{aligned}
\delta U_\mu(x) &= -agU_\mu(x)\bar{\varepsilon}(x)\gamma_\mu\lambda(x) - ag\bar{\varepsilon}(x+a\widehat{\mu})\gamma_\mu\lambda(x+a\widehat{\mu})U_\mu(x), \\
\delta U_\mu^\dagger(x) &= ag\bar{\varepsilon}(x)\gamma_\mu\lambda(x)U_\mu^\dagger(x) + agU_\mu^\dagger(x)\bar{\varepsilon}(x+a\widehat{\mu})\gamma_\mu\lambda(x+a\widehat{\mu}), \\
\delta\lambda(x) &= -\frac{i}{g}\sigma_{\rho\tau}\mathcal{G}_{\rho\tau}(x)\varepsilon(x), \\
\delta\bar{\lambda}(x) &= \frac{i}{g}\bar{\varepsilon}(x)\sigma_{\rho\tau}\mathcal{G}_{\rho\tau}(x)
\end{aligned}$$

they reduce to the continuum SUSY transformations in the limit $a \to 0$.

Numerical Simulations

In order to renormalize the SUSY WIs on the lattice a possible operator mixing has to be taken into account. X_S mixes with operators of equal or lower dimension

Bochicchio, Maiani, Martinelli, Rossi & Testa '85

- $\Delta_\mu S_\mu$
- $\Delta_\mu T_\mu = -\Delta_\mu \mathcal{G}^a_{\mu\nu}\gamma_\nu\lambda^a$ mixing current
- $\chi = \frac{1}{2}\mathcal{G}^a_{\mu\nu}(x)\sigma_{\mu\nu}\lambda^a(x)$

The renormalized SUSY WIs

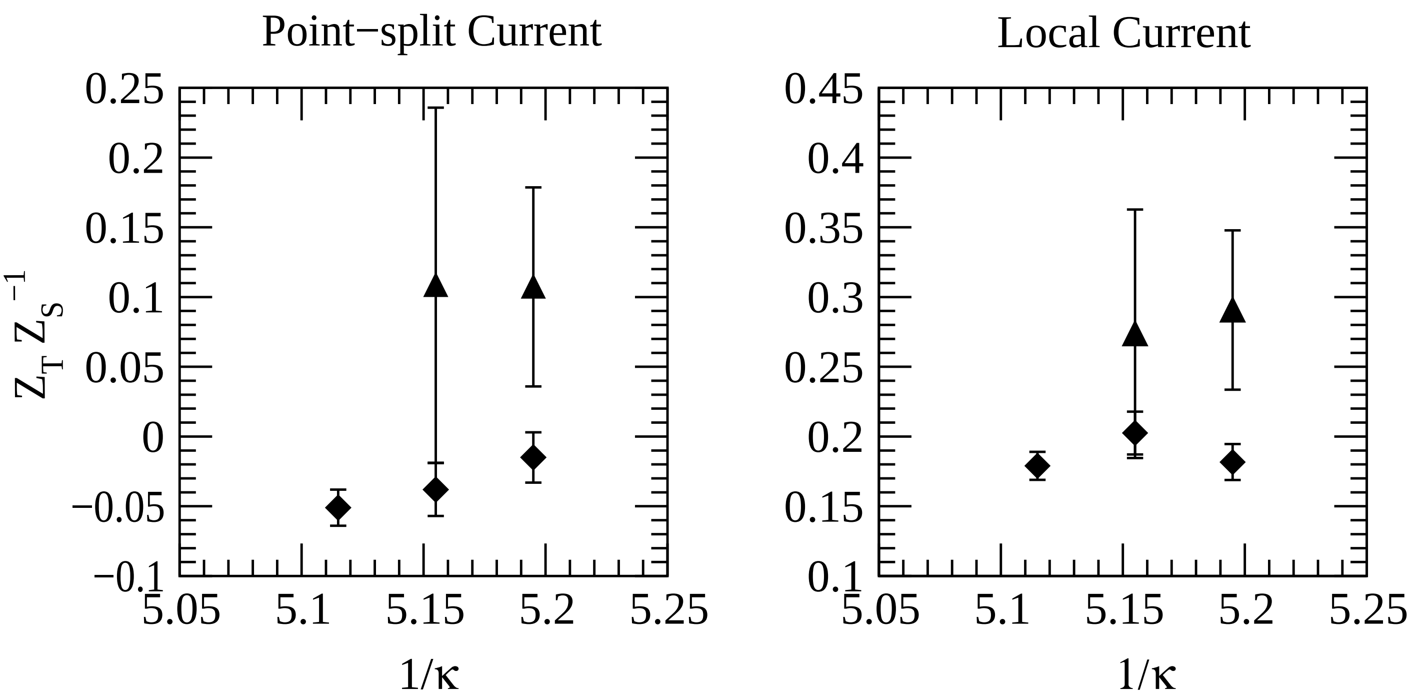
$$Z_S\langle\Delta_\mu S_\mu(x)\mathcal{O}(y)\rangle + Z_T\langle\Delta_\mu T_\mu(x)\mathcal{O}(y)\rangle = m_S\langle\chi(x)\mathcal{O}(y)\rangle + O(a)$$

where $m_S = m_0 - a^{-1}Z_\chi$.

Consider the zero spatial momentum WI obtained by the summation over the spatial coordinates with $\mathcal{O}(x) \to \bar{\mathcal{O}}^T(x) \equiv C^{-1}\mathcal{O}(x)$

$$\sum_{\vec{x}}\langle\Delta_0 S_0(x)\bar{\mathcal{O}}^T(y)\rangle + \frac{Z_T}{Z_S}\sum_{\vec{x}}\langle\Delta_0 T_0(x)\bar{\mathcal{O}}^T(y)\rangle = \frac{m_S}{Z_S}\langle\chi(x)\bar{\mathcal{O}}^T(y)\rangle + O(a)$$

Donini, Guagnelli, Hernandez & Vladikas '98



$Z_T Z_S^{-1}$ as a function of $1/k$ with the insertion operator $\chi(x)$ (filled diamonds) and $T_0^{loc}(x)$ (filled triangles). (DESY-Münster Collaboration '02)

Only for insertion $\chi(x)$, $Z_T Z_S^{-1} = -0.039(7)$ for the point-split current

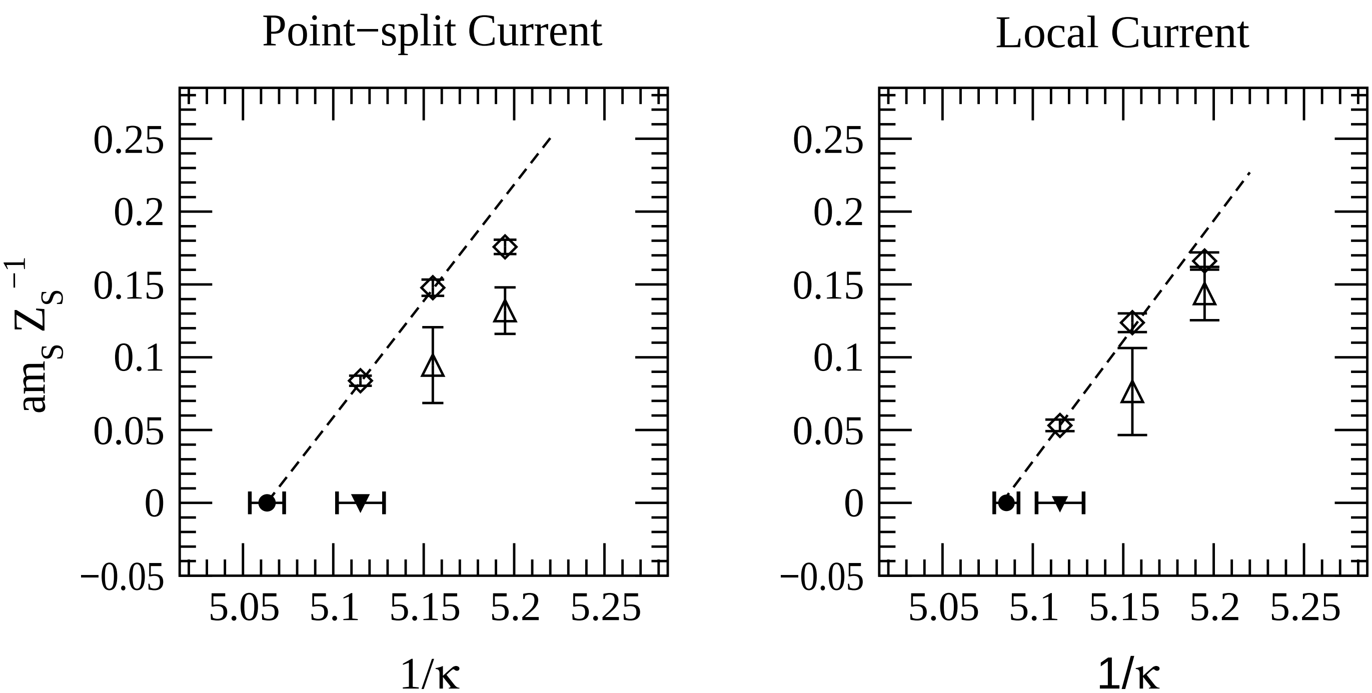
$Z_T Z_S^{-1} = 0.185(7)$ for the local current.

Surprising small values!

An estimate of $Z_T Z_S^{-1}$ for the point split current at $\beta = 2.3$ at 1-loop perturbative calculation is $Z_T Z_S^{-1} \equiv Z_T|_{1-loop} = -0.074$.

Taniguchi, '00

Also $Z_T Z_S^{-1}$ for the local current. (Preliminary results in DESY-Münster Collaboration '01)



$am_s Z_S^{-1}$ as a function of $1/k$ with the insertion operator $\chi(x)$ (diamonds) and $T_0^{loc}(x)$ (triangles). (DESY-Münster Collaboration '02). A linear extrapolation is also reported. The filled triangle indicates the determination of the k_c from the first order phase transition $k_c = (0.1955 \pm 0.0005)$.

Here the value of k_c from the SUSY WIs is $k_c = 0.19750(38)$ for the point split current and $k_c = 0.19647(27)$ for the local current.

SUSY WIs in Lattice Perturbation Theory

Things are more complicated!

In order to do perturbation theory, we have to do gauge fixing. $\rightarrow$ new terms appear in the WIs: the gauge fixing term (GF) the Fadeev-Popov-term (FP), and from the gauge variation of the involved operators (CT).

De Wit & Freedman '75

Taking into account all the contribution to the action, including GF and FP terms, the SUSY WIs reads

$$\left\langle \mathcal{O} X_S(x) \right\rangle = \left\langle \mathcal{O}\Delta_\mu S_\mu(x) - 2m_0 \mathcal{O}\chi(x) + \frac{\delta \mathcal{O}}{\delta \bar{\varepsilon}(x)}\Big|_{\varepsilon=0} - \frac{\mathcal{O}\delta S_{GF}}{\delta \bar{\varepsilon}(x)}\Big|_{\varepsilon=0} - \frac{\mathcal{O}\delta S_{FP}}{\delta \bar{\varepsilon}(x)}\Big|_{\varepsilon=0} \right\rangle .$$

Choose the operator insertion $\mathcal{O} := A^a_\alpha(y)\,\bar{\lambda}^b(z)$ non-gauge invariant.

Define a substracted $\bar{X}_S(x)$, whose expectation value is forced to vanish in the limit $a \rightarrow 0$.

$$\bar{X}_S(x) = X_S(x) + (Z_S - 1)\Delta_\mu S_\mu(x) + 2\tilde{m}\chi(x) + Z_T \Delta_\mu T_\mu(x) - \sum_i Z_i A_i .$$

- or a BRS-exact $\rightarrow A \simeq \delta_{BRS}\tilde{A}$

- or an operator that vanish using the equation of motion.

$$\langle \mathcal{O}\Delta_\mu (Z_S S_\mu(x) + Z_T T_\mu(x)) - 2(m_0 - \tilde{m})\mathcal{O}\chi(x) + \frac{\delta \mathcal{O}}{\delta \bar{\varepsilon}(x)}|_{\varepsilon=0} - \frac{\mathcal{O}\,\delta S_{GF}}{\delta \bar{\varepsilon}(x)}|_{\varepsilon=0} - \frac{\mathcal{O}\,\delta S_{FP}}{\delta \bar{\varepsilon}(x)}|_{\varepsilon=0} - \sum_i Z_i \mathcal{O} A_i \Big\rangle = 0\,.$$

Exact SUSY on the lattice

The lattice action is not unique. Improve the action in order to approach the continuum limit faster and/or have less symmetry breaking.

Improving lattice SUSY seems to be a difficult task for gauge theories because on the lattice the gauge field and the fermions are treated in a very different way.

Introduction to lattice Wess Zumino model

Dondi & Nicolai '77
Banks & Wendey '82
Bartels & Bronzand '83

It is possible to obtain perfect SUSY respect to the SUSY transformations. Nice examples:

Golterman & Petcher '89
Bietenholz '99
Catterall & Karamov '01

A perfect SUSY in $2d$ and $4d$ Wess-Zumino model, free theory, is achieved in terms of block variable renormalization group transformation (RGT). → Map a system from a fine lattice to a coarser lattice

Starting from

$$\mathcal{L} = \bar{\psi}\gamma_\mu\partial_\mu\psi + \partial_\mu\phi\partial_\mu\phi\,,$$

invariant under the tranformations

$$\delta\psi = -\gamma_\mu\partial_\mu\phi\varepsilon\,, \quad \delta\phi = \bar{\varepsilon}\psi$$

One can construct the perfect lattice action S by blocking from the continuum, which corresponds to a block variable RGT with blocking factor infinity

$$e^{S[\Psi,\Phi]} = \int D\psi D\phi e^{s[-\psi,\phi]-T[\Psi,\psi,\Phi,\phi]} \,.$$

The resulting perfect action is

$$S[\Psi,\Phi] = \frac{1}{(2\pi)^2}\int_{\pi}^{\pi} d^2p \Big\{ \bar\Psi(-p)\Delta^f(p)^{-1}\Psi(p) + \Phi(-p)\Delta^s(p)^{-1}\Phi(p)\Big\}$$

$$\begin{aligned} \Delta^f(p) &= \sum_l \frac{\prod(p+2\pi l)^2}{i(p_\mu+2\pi l_\mu)\gamma_\mu} + \alpha^f(p) \\ \Delta^s(p) &= \sum_l \frac{\prod(p+2\pi l)^2}{i(p_\mu+2\pi l_\mu)\gamma_\mu} + \alpha^s(p) \end{aligned}$$

Locality requires $\alpha^f \neq 0$ which breaks the chiral symmetry but it is still present in the observables.

Bietenholz '99

Starting from a simple discrete model with SUSY invariance

$$S = \frac{1}{2} N_i^\alpha N_i^\alpha + \bar\psi_i^\alpha M_{ij}^{\alpha\beta}\psi_j^\beta$$

how can be generalized to a $2d$ Wess-Zumino model with extended $N = 2$ SUSY. (the only possible solution).

Derive a set of exact and broken lattice WIs

$$\langle \bar\psi_i^\alpha \psi_j^\beta \rangle + \langle N_j^\beta x_i^\alpha \rangle = 0 \,.$$

Simulate the model with HMC algorithm for a range of masses and couplings.

- Weak coupling regime: it is possible to extract fermion and boson masses and to verify their equality within statistical errors. Also the WIs are satisfy to high precition.

- Strong coupling: Difficulties to extracting masses for coarse lattices (but seems to go away for finer lattices).

Catterall & Karamov '01

Ginsparg-Wilson (GW) fermions

For a review on exact lattice chiral symmetry please see plenary talk by Giusti, this conference.

Lattice Dirac operators D can be constructed which are local, do not have the doubling problem and satisfy the GW relation

$$\{D, \gamma_5\} = D\gamma_5 D \Leftrightarrow \{D^{-1}, \gamma_5\} = \gamma_5$$

Ginsparg & Wilson '82

and an explicit solution to the algebra

Neuberger '98

with quite interesting chiral properties

Lüscher '98
Hasenfratz '98

It is interesting to see what we learn of lattice SUSY using the GW algebra. Using Wess-Zumino model. All theoretical, interesting, but still far away from applications.

Fujikawa & Ishibashi '01
Fujikawa '02
Aoyama & Kikukawa '99
So & Ukita '99,'01

One difficulty to define SUSY on the lattice is the failure of the Leibniz rule. The lattice version

$$(\nabla(fg))(x) = (\nabla f)(x)g(x) + f(x)(\nabla g)(x) + a(\nabla f)(x)(\nabla g)(x)$$

Breaking of SUSY by lattice artifacts are order $O(a)$.

Leibniz rule is recovered if $|ak_\mu| < \delta$ in the limit $a \to 0$.

$\to$ all breaking terms induced by the failure of the Leibniz rule are irrelevant in the continuum limit.

Using Wess-Zumino model by applying higher derivative regularization and using GW operator.

Fujikawa '02

A conflict between the lattice chiral symmetry and the Majorana condition for Yukawa couplings when GW algebra is used.

$$S = \int \phi^\dagger \phi + \int (m\phi^2 + g\phi^3) + h.c.$$

Lattice chiral symmetry together with a naive Bose-Fermi symmetry is imposing by sacrifying the Majorana condition but preserving a SUSY-like symmetry in the free part of the lagrangian.

Fujikawa & Ishibashi '01

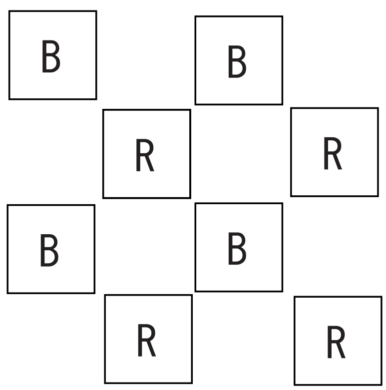
Extension of GW relation to the SUSY free theories (similar approach to Bietenholtz). Exact lattice SUSY tranformation can be defined without ambiguities. Applied to the $2d$ two chiral-multiplets as an application.

So & Ukita '99

preSUSY

A new formulation of SYM on the lattice with an exact fermionic symmetry. First it is considering the model in a fundamental lattice. $\rightarrow$ one-cell model. and deriving the preSUSY transformations. Then it is extended to the entire lattice.

The lattice action has a peculiar form (has a $2a$ translational invariance, not the usual a ones. This is called Ichimatsu lattice (similar to a chessboard)



Itoh, Kato, Sawanaka, So & Ukita '01

$\Longrightarrow$ talk by Sawanaka (study of the continuum limit)

$\Longrightarrow$ talk by So (study of the phase transition)

No exact balance between fermionic and bosonic degres of freedom (because of the staggered fermion action!)

$\Longrightarrow$ talk by Kaplan

♣ A new method for implementing SUSY on a spatial lattice in gauge theories including $N = 4$ SYM theories, with continuous Minkoswki time in a way that eliminates or reduce the problem of fine tuning. The extension to Euclidean space should not be a problem.

Start from SYM theories with low N in high dimension: $3 + 1$, $5 + 1$, $9 + 1$ and global R symmetry group

$\rightarrow$ reduced to SYM theories with extended SUSY in small dimensions $(0 + 1)$.

Start from a mother theory, SUSY QM with extended SUSY.

The spatial lattice is constructing by orbifolding by Z_N gauge and R symmetries, partially breaking the extended supersymmetry and producing N-site lattice dimensions.

Taking the continuum limit of the daughter theory result in a higher dimensional QFT with the original extended SUSY restored, together with Poincaré invariance.

- The lattice used can retain some exact SUSY's !! which facilitates the recovery of the remaining SUSY's in the continuum limit. Protected the renormalizability.

Kaplan, Katz & Unsal '02

$\Longrightarrow$ talk by Campostrini

Hamiltonian formalism

Putting a SUSY field theory on the lattice which exactly preserve a $1d$ SUSY subalgebra of the continuum $N = 1$ SUSY algebra.

(An advantage in comparison with the standard lattice formulation)

This is enough to garantee the most important properties of SUSY like pairing of positive energy states.

Beccaria, Campostrini, F. '01

Example: One of the two SUSY generators

$$Q \equiv \sum_{n=1}^{L} \left[\pi_n \psi_n^1 - \left(\frac{\phi_{n+1} - \phi_{n-1}}{2} + V(\phi) \right) \psi_n^2 \right]$$

ψ_n is a Majorana fermion with two components.

Simulation using GFMC which is very efficient in computing the ground state energy E_0 and therefore can be used to study the pattern of SUSY breaking $\rightarrow$ $E_0 > 0$. (GFMC algorithm can distinguish between 0 and 10^{-5}).

while SUSY unbroken $E_0 = 0$.

For $V(\psi) = \lambda_2 \phi^2 + \lambda_0\,, \quad \lambda_2 > 0$

- Predictions of strong coupling (SUSY breaking) are different from predictions of weak coupling (SUSY breaking when $\lambda_0 > 0$).

- Simulations conferms predictions of the strong coupling up to large $\xi \to \infty$, altough extrapolation to the continuum limits need further work.

Considering a $N = 1$ SUSY CS theory in $2+1$ dimensions.

$$\mathcal{L} = \text{Tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\gamma_\mu D^\mu \Psi + 2\bar{\Psi}\Psi + \frac{k}{2}\varepsilon\mu\nu\lambda\Big[A_\mu \partial_\nu A_\lambda + \frac{2i}{3} g A_\mu A_\nu A_\lambda\Big]\Big)$$

Used SDLCQ (Supersymmetric Discrete Light-Cone Quantization) which is a numerical Hamiltonian method that can be used to solve any theory with enough SUSY to be finite.

Discretizing the theory by imposing periodic boundary conditions on the boson and fermion field in terms of discrete momentum modes $k = nP/K$ which depends on

- K positive integer that determines the resolution.

Hiller, Pinsky & Trittman

(hep-th/0203162, hep-th/0112151, hep-th/0206197)
The low-energy spectrum mass using the fit $M^2 = M_\infty^2 + b(1/K)$

Find out approximate BPS states which have non-zero masses.

$\Longrightarrow$ talk by Trittman (numerical test of the Maldacena conjecture)

$\Longrightarrow$ talk by Pinsky

$\Longrightarrow$ talk by Wosiek

- Face a problem of a cut-off which violates SUSY and observe restoration of SUSY when it is taken away i.e. to the infinity.

- Applied to
  - Wess-Zumino QM
  - SYMQM in $2d$ and $4d$. (no sign problems is find out).
  - Complete spectrum, Witten index and identification of SUSY multiplets.

Wosiek '02

Conclusions & Outlook

Until now a big effort has been made it order to describe SUSY on the lattice. But practical results are limited only to some $1+1$ dimensional theories and $N = 1$ SYM theory in $3+1$ dimensions.

Wilson fermions & DWF
Promising results!

(specially the lattice SUSY WIs for Wilson fermions).

- Study of the phase transition for bigger lattices. (Crossover?)

- Spectrum of masses in bigger lattices and near the critical point.

.... Exact lattice SUSY?

... We need more theoretical inputs!

- Physicist should explore other theories beyond the real-world QCD and placing QCD in a wider variety of theories may become a powerful tool for understanding it.