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LETTER TO THE EDITOR

The spherical model on graphs

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Abstract. We extend the lattice spherical model of Berlin and Kac to infinite graphs (describing inhomogeneous structures such as fractals, polymers and amorphous materials). We analytically calculate the exact values of the critical exponents, which turn out to depend only on the vibrational spectral dimension $\bar{d}$ of the graph. This functional dependence coincides with the analytic continuation in d of the corresponding exponents for the lattice model. This result provides an example of geometrical universality classes for non-translationally invariant systems and strongly suggests considering $\bar{d}$ as the natural generalization of the Euclidean dimension d for critical phenomena.

The study of statistical models on lattices has brought out the fundamental role played by the spatial dimension d in phase transitions and critical phenomena. Indeed, it is well established that the possibility of spontaneous symmetry breaking depends on the lattice geometry only through d and it is known that the critical exponents are determined only by d and the symmetry of the Hamiltonian. However, in spite of these fundamental and general results, only a few models have been solved exactly in their critical regime and usually exact solutions are not available for $d > 2$. Only the spherical model on lattices has been solved exactly in any dimension, allowing one to know the dependence of its critical exponents on d [1]. This model is even more interesting for two additional reasons: it has been shown to be the limit for $n \to \infty$ of classical Heisenberg O(n) spin models [2] and its critical exponent has an analytic continuation for real values of d for $2 < d < 4$. The former result is the basis for the well known $1/n$ expansion for Heisenberg models, while the latter gives rise to very intriguing questions about the physical meaning of non-integer dimensions. In fact non-integer dimensions are needed when dealing with non-crystalline structures (such as amorphous materials, polymers and fractals etc), where the lack of translational invariance prevents one from using such useful concepts and techniques as reciprocal lattices and Fourier transform. However, although several definitions of generalized dimensions have been proposed, it is not yet clear which is the right one (if any) to describe critical phenomena. At present the best candidate for such a role is the spectral dimension $\bar{d}$, introduced by Alexander and Orbach [4] to describe the vibrational spectrum of fractal structures and by Dhar [3] in connection with the infrared singularities of the Gaussian model on a class of generalized networks. Indeed $\bar{d}$ turns out to rule not only the vibrational spectrum, but also the long time average behaviour of random walks, the free

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energy singularities of the Gaussian model [5] and the possibility of continuous symmetry breaking [6]. Still, up to now the lack of exactly solved models showing phase transitions on non-translationally invariant structures has prevented major progress in understanding the relevance of $\bar{d}$.

Here we generalize the spherical model to graphs, i.e. to generic networks not necessarily translation invariant, we show that it has a phase transition at $T > 0$ if $\bar{d} > 2$ and we calculate its thermodynamical critical exponents exactly for any $\bar{d}$. These exponents are expressed as simple functions of $\bar{d}$ only. Moreover, for $2 < \bar{d} < 4$ they do coincide with the analytic continuation of the corresponding ones on lattices and for $\bar{d} > 4$ they are the usual mean-field exponents found for $d > 4$. All these results strongly confirm the fundamental role played by $\bar{d}$ in critical phenomena. In this paper we first introduce the basic ideas concerning graphs and the spectral dimension and its relation with the Gaussian model and we recall the spherical model on lattices together with its solution. Then we introduce the generalized spherical model on graphs and we exactly solve its critical regime by mapping it into the critical regime of a suitable Gaussian model. Finally, we give the critical exponents as functions of $\bar{d}$ and discuss the results.

A graph G is a set of N points ($N \to \infty$ in the thermodynamic limit) with a topology described by its adjacency matrix A_{ij}, whose elements are equal to 1 if the points i and j are nearest neighbours and equal to 0 in all other cases. The usual physical way of defining the spectral dimension $\bar{d}$ consists in considering G as an oscillating network with masses m at each point, connected along the links by springs with elastic constant k. The spectral dimension $\bar{d}$ is then defined by the asymptotic behaviour of the density of modes with frequency ω: if $\rho_\omega(\omega) \sim \omega^k$ for $\omega \to 0$, then $\bar{d} \equiv k + 1$. From a rigorous mathematical point of view, $\bar{d}$ can be defined by considering the spectrum of the Laplacian operator on G, defined by $L_{ij} \equiv z_i \delta_{ij} - A_{ij}$, $z_i = \sum_j A_{ij}$ being the coordination number of site i. It can be shown that the leading asymptotic behaviour of the spectral density $\rho_l(l)$ of L for $l \to 0^+$ cannot be slower or faster than a power law if the following conditions are fulfilled: (1) $z_i \leqslant z_{\max} < \infty$ for every i; (2) calling $N_i(r)$ the number of point G at a chemical distance $\leqslant r$ from i, it is possible to find two positive constants a and b such that $N_i(r) < ar^b$ for every i. These conditions are fulfilled by graphs usually introduced to describe real physical systems, such as fractals, bundled graphs and quasicrystals. Indeed, for all these graphs, when it has been possible to calculate it exactly, $\rho_l(l)$ turns out to behave exactly as a power law. In this case $\bar{d}$ can be defined by $\rho_l(l) \sim l^{\bar{d}/2-1}$ for $l \to 0$.

The Gaussian model on G is defined by the Hamiltonian:

$$H = \tfrac{1}{2} \sum_{ij} \phi_i (J L_{ij} + m_i^2 \delta_{ij}) \phi_j - h \sum_i \phi_i \qquad (1)$$

where ϕ_i is a real field, $J > 0$ is a ferromagnetic coupling, h an external magnetic field and $m_i^2 = \alpha_i m^2$, with $1/K < \alpha_i < K$ for some positive K [7]. Its specific free energy f_G is given by

$$f_G(J, m_i^2, h) = \lim_{N\to\infty} \frac{1}{N} F = - \lim_{N\to\infty} \frac{1}{N} \log Z \qquad (2)$$

where Z is the partition function calculated according to the Boltzmann weight $\exp(-H)$. The spectral dimension is related to the singular part of f_G for $h = 0$ and $m^2 \to 0$ by:

$$\text{Sing } (f) \sim (m^2)^{\bar{d}/2}. \qquad (3)$$

The spherical model introduced by Berlin and Kac [1] is defined on an N-site lattice by the ferromagnetic Hamiltonian for the real field ϕ_i

$$H = -J \sum_{i,j=1}^{N} \phi_i A_{ij} \phi_j - h \sum_{i=1}^{N} \phi_i \tag{4}$$

with the *spherical constraint* $\sum_i \phi_i^2 = N$, where h represents an external field. The partition function is

$$\mathcal{Z} \equiv \int_{-\infty}^{\infty} \prod_{i=1}^{N} \mathrm{d}\phi_i \, \mathrm{e}^{-\beta H} \delta\left(N - \sum_i \phi_i^2\right). \tag{5}$$

Using the integral representation of the δ-function

$$\delta(x) = \frac{1}{2\pi \mathrm{i}} \int_{\alpha - i\infty}^{\alpha + i\infty} \mathrm{d}\sigma \, \mathrm{e}^{\sigma x} \tag{6}$$

the partition function becomes

$$\mathcal{Z} = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \mathrm{d}\phi_i \int_{\alpha - i\infty}^{\alpha + i\infty} \mathrm{d}\sigma \, \mathrm{e}^{N\sigma - \mathcal{S}_N(\phi_i, \sigma)} \tag{7}$$

where

$$\mathcal{S}_N(\phi_i, \sigma) \equiv - \sum_{i,j=1}^{N} \phi_i (\sigma \delta_{ij} - \beta A_{ij}) \phi_j - \beta h \sum_{i=1}^{N} \phi_i. \tag{8}$$

Under the condition of positivity of the matrix $\sigma I - \beta A$, (i.e. $\mathrm{Re}\,(\sigma) = \alpha > \beta z$) the integration order can be inverted and the partition function can be expressed as a superposition of Gaussian integrals. In this form the calculation can be easily done performing a linear unitary transformation which diagonalizes the matrix. Due to the translational invariance of the lattice this transformation is the Fourier transform of the field ϕ_i.

In order to investigate the existence of phase transitions the model is studied in the thermodynamic limit ($N \to \infty$) with zero external field. It turns out that the partition function is analytic at any temperature on lattices with $d \leqslant 2$, whereas it presents a singular behaviour below a certain critical temperature T_c on lattices with $d \geqslant 3$. This implies the presence of a phase transition. The only relevant parameter in the critical behaviour is therefore the dimension of the lattice. The critical exponents are calculated exactly for any integer value of the dimension d and they can be formally determined for real values of the parameter d through analytic continuation.

The spherical model can be defined on a generic graph through the Hamiltonian (4) with the *generalized spherical constraint* $\sum_i z_i \phi_i^2 = N$. We assume the coordination numbers to be bound: $1 \leqslant z_i \leqslant z_{\max}$. On a regular lattice this constraint is exactly equivalent to the usual one. On the other hand, on a generic graph this is the only generalization which preserves the definiteness of the model itself at any T. Using the integral representation (6) the partition function can be written in the form (7) with

$$\mathcal{S}_N(\phi_i, \sigma) \equiv - \sum_{i,j=1}^{N} \phi_i (\sigma z_i - \beta A_{ij}) \phi_j - h \sum_{i=1}^{N} \phi_i \tag{9}$$

where the external field has been rescaled by a factor β.

If the matrix $(\sigma Z - \beta A)$, where $Z \equiv \mathrm{diag}(z_i)$, has positive eigenvalues, i.e. if $\alpha > \beta$, the partition function of the spherical model is

$$\mathcal{Z} = \frac{1}{2\pi \mathrm{i}} \int_{\alpha - i\infty}^{\alpha + i\infty} \mathrm{d}\sigma \, \mathrm{e}^{N\sigma} \mathcal{Z}_{\mathcal{G}}(\beta, \sigma, h) \tag{10}$$

where $\mathcal{Z}_\mathcal{G}$ is the partition function of the Gaussian model with squared masses $m_i^2 \equiv (\sigma - \beta) z_i$, $m^2 \equiv \sigma - \beta$, $\alpha_i \equiv z_i$, and couplings $J = \beta$:

$$\mathcal{Z}_\mathcal{G}(\beta, \sigma) \equiv \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \mathrm{d}\phi_i \, \mathrm{e}^{-\mathcal{S}_N(\phi_i, \sigma)}. \tag{11}$$

The Gaussian free energy per particle is easily evaluated:

$$\begin{aligned} g(\beta, \sigma, h) &\equiv f_G(\beta, (\sigma - \beta) z_i, h) \equiv \frac{1}{N} \ln \mathcal{Z}_\mathcal{G}(\beta, \sigma, h) \\ &= -\frac{1}{2N} \operatorname{Tr} \ln(\sigma Z - \beta A) + \frac{h^2}{N} \sum_{i,j=1}^{N} (\sigma Z - \beta A)^{-1}_{ij}. \end{aligned} \tag{12}$$

In the thermodynamic limit $N \to \infty$ the saddle point method can be used yielding $\mathcal{Z} \propto \exp\{N[s + g(\beta, s, h)]\}$ where $s = s(\beta, h)$ is the solution of the saddle point equation

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} g(\beta, \sigma, h)\bigg|_{\sigma = s} + 1 = 0. \tag{13}$$

This equation contains the whole physical information on the system since it implicitly expresses the parameter s as a function of temperature and magnetic field. In general, it is impossible to explicitly solve the saddle point equation because of the lack of translational invariance. Indeed, this implies that the matrix $(\sigma Z - \beta A)$ cannot be diagonalized by a Fourier transform. The diagonalizing transformation strongly depends on the geometrical properties of the graph and is in general unknown. Differentiating expression (12) and observing that Z and A commute under trace, the saddle point equation yields, after some algebra,

$$\lim_{N \to \infty} \frac{1}{2N} \sum_{i=1}^{N} C_{ii}(s) z_i - \frac{h^2}{2N} \frac{\mathrm{d}}{\mathrm{d}\sigma} \sum_{i,j=1}^{N} C_{ij}(\sigma)\bigg|_{\sigma = s} = 1 \tag{14}$$

where C_{ij} are the correlation functions of the Gaussian model for $h = 0$

$$C_{ij}(\sigma) = [(\sigma Z - \beta A)^{-1}]_{ij}. \tag{15}$$

Recalling the relevant property of the correlation functions [5] $\sum_j C_{ij} m_j^2 = 1$ and the choice of bounded coordination numbers we have

$$\frac{1}{2} \bar{C}(s) + \frac{h^2}{\zeta(s)(s - \beta)^2} = 1 \tag{16}$$

where $\bar{C}(s) \equiv \lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} C_{ii}(s) z_i = \partial f_G / \partial m^2$ and $\zeta(s)$ is a smooth function bounded between 1 and $z_{\max}$ for any value of s. The existence of phase transition for the spherical model now depends on the existence of real solutions for equation (16). The general solution is obtained studying the asymptotic behaviour of the average correlation function $\bar{C}(s)$ near its infrared singularity $m_i^2 \to 0$, i.e. $s \to \beta$:

$$\operatorname{sing} \bar{C}(s) \sim (s - \beta)^{\frac{\bar{d}}{2} - 1}. \tag{17}$$

We find that the partition function of the spherical model on a graph is an analytic function of the temperature everywhere if an external magnetic field is present. If the field h is switched off, the behaviour of the model depends deeply on the large scale geometry of the graph. This means that if the graph is *transient*, i.e. if $\bar{d} > 2$, there exists a critical point T_c. The partition function has a cut in the (T, h) plane along the temperature axis from zero to T_c. On the other hand, if the graph is *recursive*, i.e. if $\bar{d} \leqslant 2$, the model does not present phase transitions.

The critical exponents for the spherical model on a graph of spectral dimension $\bar{d}$ can be calculated exactly through asymptotic expansions near the critical point ($T = T_c$, $h = 0$).

Setting $T = T_c$ we find the average magnetization per site $m(h)$ for $h \to 0$. Differentiating with respect to h the specific free energy $f(s) = s + g(s)$ we find

$$m(h) \propto \frac{h}{\epsilon} \tag{18}$$

where $\epsilon \equiv (1 - \beta/s) \to 0$ near the critical point. The asymptotic dependence of ϵ on h is found solving the saddle point equation (16) for small ϵ and introducing the infrared singular behaviour of the $\bar{C}(s)$. Substituting the result in (18) we are able to find the critical exponent δ defined by $m(h) \sim h^{1/\delta}$ for $T = T_c$ and $h \sim 0$. We find $\delta = (\bar{d} + 2)/(\bar{d} - 2)$ if $2 < \bar{d} < 4$ and $\delta = 3$ if $\bar{d} > 4$. For $\bar{d} = 4$ $m(h)$ behaves as $[h \ln(1/h)]^{1/3}$.

To study the critical behaviour in the disordered phase ($T \to T_c^+$) we directly set $h = 0$. This is possible because in this region $\mathcal{Z}(h, \beta)$ is analytic and the limits $h \to 0$ and $T \to T_c^+$ can be interchanged. Both the critical exponents γ and α, can be calculated through analytic expansions for $\epsilon \to 0^+$. For the susceptibility we have $\chi \propto \frac{1}{\epsilon}$. Using equation (16) we express ϵ as a function of the reduced temperature $t \equiv (T - T_c)/T_c$. Applying the definition of γ : $\chi \sim t^{-\gamma}$ for $t \to 0^+$, we find $\gamma = 2/(\bar{d} - 2)$ for $2 < \bar{d} < 4$, while $\gamma = 1$ when $\bar{d} > 4$. For $\bar{d} = 4$ the critical behaviour of the magnetic susceptibility is given by $\chi \sim t^{-1} \ln t^{-1}$. In order to evaluate the critical behaviour of the specific heat c we use the following asymptotic relation [8] between susceptibility and specific heat which remains valid until $T > T_c$:

$$c = \frac{1}{2} K_B + \frac{1}{2\zeta(\beta)} \chi^{-2} \frac{\mathrm{d}\chi}{\mathrm{d}T}. \tag{19}$$

The critical exponent α, defined by $c_{\text{sing}} \sim t^{-\alpha}$ for $t \to 0^+$, is $\alpha = (\bar{d} - 4)/(\bar{d} - 2)$ if $2 < \bar{d} < 4$. Notice that $\alpha < 0$ implies $c_{\text{sing}} = 0$ at T_c and therefore $c = (\frac{1}{2})K_B$. For $\bar{d} > 4$ we find $\alpha = 0$. This means that c has two different finite limits for $t \to 0^{\pm}$. For $\bar{d} = 4$ we find $c \sim c_1 + c_2[\ln(t)^{-1}]^{-1}$, where c_1 and c_2 depend on the detailed structure of the graph.

In the phase of broken symmetry the saddle point equation has no solution for $h = 0$. Therefore, we have to study the two limits $h \to 0^{\pm}$. For $T < T_c$ we have $\epsilon \to 0^+$. Solving (16) with respect to ϵ and substituting in equation (18), we find $m(T) \propto \sqrt{-t}\,\text{sgn}\,(h)$, which gives $\beta = \frac{1}{2}$. This result is independent of $\bar{d}$ and holds for any graph. We emphasize that the expression of $m(T)$ is exact at any $T \leqslant T_c$. Taking the second derivative of the free energy with respect to h we find $\chi(h, \epsilon(h))$. To obtain $\chi \equiv \chi(h \to 0)$ we find, from the saddle point equation, ϵ as a function of h. If $2 < \bar{d} \leqslant 4$, χ diverges for $h \to 0$ and γ' cannot be defined. If $\bar{d} > 4$ the $h \to 0$ limit of $\chi(h)$ exists and the critical exponent γ' defined by $\chi \sim (-t)^{-\gamma'}$, $t \to 0^-$, is equal to γ, i.e. $\gamma' = 1$. In a similar way we can show that the specific heat in the ordered phase is $c = (\frac{1}{2})K_B$. This result is exact at any $T < T_c$ and for any transient graph.

In a similar way the spherical model can also be solved on recursive graphs, i.e. graphs with $\bar{d} \leqslant 2$. In this case the saddle point equation has a real solution $s(h, T)$ for any h and T except for $T = 0,\ h = 0$. However the thermodynamic functions show a critical behaviour for $T,\ h \to 0$ since in this limit $s \to \beta$, i.e. the infrared singularity of $\bar{C}(s)$ comes into play. Actually, we find that the response functions diverge as power laws if $1 \leqslant \bar{d} < 2$ while they behave exponentially in T if $\bar{d} = 2$. Therefore, if $\bar{d} < 2$ we have a critical behaviour described by true critical exponents. For $\epsilon \to 0^+$ we find that when $\bar{d} < 2$ the magnetic susceptibility at low temperature is $\chi \sim T^{2/(\bar{d}-2)}$ yielding a critical exponent $\gamma = -2/(\bar{d} - 2)$. If $\bar{d} = 2$ we have $\chi \sim \exp(-\tau/T)$ where τ is a graph dependent parameter. Since we are in an analyticity region, the relation (19) applies, yielding $c = (\frac{1}{2})K_B + c_{\text{sing}}$, where $c_{\text{sing}} \sim T^{-\bar{d}/(\bar{d}-2)}$ if $\bar{d} < 2$, i.e. $\alpha = \bar{d}/(\bar{d} - 2)$, and $c_{\text{sing}} \sim T^{-2} \exp(-\tau/T)$ if $\bar{d} = 2$. If we set $T = 0$ we find that m is finite for $h \to 0$ for any $\bar{d} \leqslant 2$ formally yielding $\delta = \infty$. The values of all the critical exponents of the spherical model on a graph are summarized in table 1.

Table 1. Critical exponents of the spherical model on a graph of spectral dimension $\bar{d}$.

	$1 \leqslant \bar{d} < 2$	$2 < \bar{d} < 4$	$\bar{d} > 4$
$T = T_c$	$\delta \to \infty$	$\delta = \frac{\bar{d}+2}{\bar{d}-2}$	$\delta = 3$
$T < T_c$	—	γ' does not exist	$\gamma' = 1$
$T > T_c$	$\gamma = \frac{2}{\bar{d}-2}$	$\gamma = \frac{2}{2-\bar{d}}$	$\gamma = 1$
$T < T_c$	—	$c = \frac{1}{2} K_B$	$c = \frac{1}{2} K_B$
$T > T_c$	$\alpha = \frac{\bar{d}}{\bar{d}-2}$	$\alpha = \frac{\bar{d}-4}{\bar{d}-2}$	$\alpha = 0$
$T < T_c$	—	$\beta = \frac{1}{2}$	$\beta = \frac{1}{2}$

As we previously mentioned, all critical exponents depend only on $\bar{d}$. This is a strong evidence of universality. Indeed, the spectral dimension itself has deep universal properties being related only to large scale geometry, regardless of local topological details. It can be shown [9] that $\bar{d}$ is left unchanged under the addiction of ferromagnetic couplings up to kth neighbours, with finite k. Moreover, the critical exponents are still the same for a generic distribution of bounded ferromagnetic couplings $J_{ij} \equiv g_{ij} A_{ij}$, with $1/K < g_{ij} < K$, together with the generalized spherical constraint $\sum_i J_i \phi_i^2 = N$, where $J_i \equiv \sum_j J_{ij}$. All these properties clearly also hold for regular lattices, since they are particular cases of graphs, proving the invariance of the usual critical regime under the introduction of ferromagnetic disorder and finite range geometrical disorder.

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